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POLYNOMIALS ON SP-RING

K. Shanmuga Priya[1] and M. Mullai[2]
[1] Research Scholar, premsathya79@gmail.com
[2] Assistant Professor, mullaim@alagappauniversity.ac.in

Abstract

Algebra acts as one of the building block for mathematics today. In this paper, the concept of Polynomials on SP-Ring has been introduced. The definition of SP-Polynomial, degree of SP-polynomial, primitive SP-Polynomial, irreducible SP-Polynomial, content of SP-Polynomial, some theorems, Gauss lemma and the Eisenstein Criterion are also defined and established.

Keywords: SP-Polynomial, Primitive, Content, Gauss lemma, Eisenstein criterion..
Subject Classification: 2010 Mathematics Subject Classication: 08A30, 08A35, 08A40.

1 Introduction

In 1978, K.Iseki and S.Tanaka introduced the concept of BCK-Algebra[2]. K.Iseki introduced BCI-Algebra[1] in 1980. M.Mullai and K.Shanmuga priya introduced a new notion of algebra known as SP-Algebra[4], which is a subclass of BH-Algebra[3]. Further that paper was extended to SP-Ring[5] by them. In this paper, SP-Ring concept is applied to ’polynomials formed from SP-Ring. This paper display definitions, theorems related to SP-polynomials and also provide SP-Ring over unique factorization domain, Gauss lemma and Eisenstein Criterion are also established.

2 preliminaries

Definition 2.1 *An Algebra(X, $*$, e) of type (2,0) is said to be Semi SP-Algebra if*

1. $x * x = e$.
2. $x * e = x$.
3. *if* $x * y = e$ *and* $y * x = e$, *then* $x = y$, *where* $*$ *is a binary operation and e is any constant.*

Definition 2.2 *[5] A non-empty set X together with two binary operations ‘$*$’ and ‘Δ’, is called SP-Ring, if it satisfies the following axioms:*

1. *(X, $*$) is a Semi SP-Algebra.*
2. *Δ’ is associative on X.*
3. $a\Delta(b*c) = (a\Delta b)*(a\Delta c)$.

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4. $(a*b)\Delta c = (a\Delta c)*(b\Delta c)$, $\forall a, b, c \in X$.

Definition 2.3 *[5] A SP-Ring (X, ∗, Δ) is called integral domain if it has a constant e' and the set of all elements satisfying $x\,\Delta\,x = e'$ is a semi SP-Algebra under Δ.*

Definition 2.4 *[5] Let (X, ∗, Δ) be an abelian SP-Ring and $a, b \in X$, $a = 0$. Then a divides b [write a/b], if there exists an element $c \in X$ such that $b = a\,\Delta\,c$.*

Definition 2.5 *[5] In an Euclidean domain (X, ∗, Δ), an element $a \in X$ is said to be prime if 'a' cannot be expressed as $a = b\,\Delta\,c$, where $b, c = e' \in X$, and e' is constant in X corresponding to Δ.*

2.1 SP-POLYNOMIAL

Definition 2.6 *A non-empty subset S of a SP-Ring (R, ∗, Δ) is called a SP-Subring if S itself is a SP-Ring under the same operation as in R.*

Example 2.7 *$(2Z^+ \cup \{0\}, \ominus, .)$ is a SP-Subring of $(Z^+ \cup \{0\}, \ominus, .)$*

Definition 2.8 *Let R be a SP-Ring. A polynomial f(x) with coefficient in R is defined to be an expression of the form $a_0 + a_1x + a_2 x^2 + \ldots + a_n x^n$, where n is a positive integer and $a_0, a_1, \ldots a_n \in R$. a_r is called the co-efficient of x^r. $a_r x^r$ is called a term of the polynomial. This f(x) can be called as SP-Polynomial.*

The set of all polynomials with co-efficient in R is denoted by R[x].

Definition 2.9 *Let $f(x) = a_0 + a_1x + a_2 x^2 + \ldots + a_n x^n$, $a_n \neq 0$ be a non-zero polynomial in R[x]. Then the degree of f(x) written as deg f(x) is n and a_n is called the leading coefficient of f(x).*

Result 2.10 *Every SP-Ring (R, ∗, Δ) is closed under addition*

Definition 2.11 *If $p(x) = a_0 + a_1x + a_2 x^2 + \ldots + a_n x^n$ and $q(x) = b_0 + b_1x + b_2 x^2 + \ldots + b_m x^m$ are both in R[x], then $p(x) * q(x) = c_0 + c_1x + c_2 x^2 + \ldots + c_t x^t$, where for each i, $c_i = (a_i * b_i)$, each $c_i \in R$. Therefore $p(x) * q(x) \in R[x]$.*

Definition 2.12 *If $f(x) = a_0 + a_1x + a_2 x^2 + \ldots + a_m x^m$, and $g(x) = b_0 + b_1x + b_2 x^2 + \ldots + b_n x^n$ in R[x], then $f(x)\,\Delta\,g(x) = c_0 + c_1x + c_2 x^2 + \ldots + c_k x^k$, where $c_r = a_r b_0 + a_{r-1} b_1 + \ldots + a_0 b_r \in R$. Therefore $f(x)\,\Delta\,g(x) \in R[x]$.*

Theorem 2.13 *Let R be any SP-Ring. Then R[x] with ∗ and Δ defined above is also SP-Ring.*
Proof:
Let f(x) and g(x) $\in R[x]$

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Clearly $f(x) * g(x)$ *and* $f(x)\, \Delta\, g(x) \in R[x]$
Let $0 \in R[x]$ *is a constant element with respect to* $*$
Clearly $f(x) * f(x) = 0$ *and* $f(x) * 0 = f(x)$
If $f(x) * g(x) = 0$ *and* $g(x) * f(x) = 0$
To prove $f(x) = g(x)$
Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n, a_n \neq 0,$
$g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^m, b_m \neq 0,$
Then $f(x) * g(x) = c_0 + c_1x + c_2x^2 + \ldots + c_tx^t$, *where each* $c_i = (a_i - b_i)$
$g(x) * f(x) = r_0 + r_1x + r_2x^2 + \ldots + r_tx^t$, *where* $r_i = (b_i * a_i)$
Since $f(x) * g(x) = 0 = g(x) * f(x)$
$\Rightarrow f(x) * g(x) = g(x) * f(x)$
Therefore $c_0 + c_1x + c_2x^2 + \ldots + c_tx^t = r_0 + r_1x + r_2x^2 + \ldots + r_tx^t$
By definition each $c_i = r_i$
Therefore $a_i * b_i = b_i * a_i$
$\Rightarrow 2a_i * 2b_i = 0$, *since* $x * x = e$
$\Rightarrow 2(a_i * b_i) = 0$
$\Rightarrow a_i = b_i$, *Since* $x * y = 0$, *then* $x = y$
$\Rightarrow f(x) = g(x)$
Hence $R[x]$ *is a SP-Algebra under* $*$ *and* Δ
Since associative and distributive are obvious

$(R[x], *, \Delta)$ *is a SP-Ring.*

Theorem 2.14 *R[x] is an integral domain iff R is an integral domain.*
Proof: Assume R is an integral domain
To prove R[x] is an integral domain
ie) To prove R[x] has a constant e' *and the set of all elements satisfying* $f(x)\, \Delta\, f(x) = e'$ *is a SP-Algebra under* Δ
Since R is integral domain, $e' \in R$
Clearly $e' \in R[x]$ *and* $f(x)\, \Delta\, e' = f(x)$
$S = f(x)/ f(x)\, \Delta\, f(x) = e'$ *such that* $f(x) \in R[x]$
It is enough to prove that if $f(x)\, \Delta\, g(x) = e'$ *and* $g(x)\, \Delta\, f(x) = e'$, *then* $f(x) = g(x)$

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Since $f(x)\,\Delta\,f(x) = e'$, $f(x)\,\Delta\,g(x) = e'$,
$f(x)\,\Delta\,g(x) = f(x)\,\Delta\,f(x)$, $g(x)\,\Delta\,f(x) = f(x)\,\Delta\,g(x)$
$\Rightarrow f(x)\,\Delta\,g(x) = f(x)\,\Delta\,f(x)$
$\Rightarrow (f(x)\,\Delta\,g(x))\,\Delta\,g(x) = (g(x)\,\Delta\,g(x))\,\Delta\,g(x)$
$\Rightarrow f(x)\,\Delta\,(g(x)\,\Delta\,g(x)) = g(x)\,\Delta\,(g(x)\,\Delta\,g(x))$
$\Rightarrow f(x)\,\Delta\,e' = g(x)\,\Delta\,e'$
$\Rightarrow f(x) = g(x)$
Therefore R[x] is SP-Algebra under Δ
Therefore R[x] is an integral domain
Conversly
R[x] is an integral domain
To prove that R is integral domain
Since R is a SP-Subring of R[x] and e' *in R*

By definition, R is an Integral domain.

2.2 SP-POLYNOMIAL OVER UNIQUE FACTORIZATION DOMAIN

Definition 2.15 *Let R be a Unique facorization domain. Let* $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $a_n \neq 0$ *in R[x]. f(x) is said to be primitive if the g.c.d of the co-efficient of f(x) is constant* $e' \in R$.

Definition 2.16 *The g.c.d of* $a_0, a_1, \ldots a_n$ *is called the content of f(x) and is denoted by c(f).*

Theorem 2.17 *Let R be a Unique factorization domain.Let* $f(x) \in R[x]$ *be a non-constant polynomial. Then f(x) can be written as* $f(x) = c\,\Delta\,f_1(x)$ *where* $c = c(f)$ *and* $f_1(x) \in R[x]$ *is primitive.*

Proof:
Given R be a unique factorization domain
Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $a_n \neq 0$ *in R[x]*
Let c be the g.c.d of co-efficients of $a_0, a_1, \ldots a_n$
Therefore c/a_i, *for all i*
$\Rightarrow a_i = c\,\Delta\,b_i$, *where* $b_i \in R$

$$\text{Hence} f(x) = (c\Delta b_0)+(c\Delta b_1)x+\ldots+(c\Delta b_n)x^n \quad (1)$$
$$= c\Delta(b_0+b_1x+\ldots+b_nx^n)$$

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$= c\Delta f_1(x)$

Since the g.c.d of $a_0, a_1, ...a_n$ is c, the g.c.d of $b_0, b_1, ...b_n$ is a constant e' in R

Therefore $f_1(x)$ is a primitive polynomial in R[x].

Theorem 2.18 *Let R be a unique factorization domain. Then the product of two primitive polynomials in R[x] is again a primitive polynomial in R[x].*

Proof:

Given R is an Unique factorization domain

Let $f(x) = a_0 + a_1x + a_2 x^2 +...+a_n x^n, a_n \neq 0$

$g(x) = b_0 + b_1x + b_2 x^2 +...+b_m x^m, b_m \neq 0$ in R[x]

Let p be a prime element in R

Since f(x) and g(x) are primitive polynomials p does not divide all a_i ' s and b_j ' s, where i = 1, 2,...n, j = 1,2,...m

Let a_r be the first co-efficient of f(x) not divisible by p

Therefore $p/a_0, p/a_1, ...p/a_{r-1}$, p does not divide a_r

Similarly b_s is the first co-efficient of g(x) not divisible by p

$p/b_0, p/b_1, ...p/b_{s-1}$, p does not divide b_s

The co-efficient of x^{r+s} in $f(x) \Delta g(x)$ is given by $c_{r+s} = (a_0 \Delta b_{r+s} + a_1 \Delta b_{r+s-1} + ...+a_{r-1} \Delta b_{s+1}) + (a_r \Delta b_s) + (a_{r+1} \Delta b_s) + (a_{r+1} \Delta b_{s-1} +...+ a_{r+s} \Delta b_0)$

P divides $(a_0 \Delta b_{r+s} +....+a_{r-1} \Delta b_{s+1})$ p divides $(a_{r+1} \Delta b_s) + (a_{r+1} \Delta b_{s-1} +...+ a_{r+s} \Delta b_0)$ and p does not divide $a_r \Delta b_s$

$\Rightarrow$ p does not divide c_{r+s}

Therefore for any prime element p in R there exist some co-efficient of $f(x) \Delta g(x)$ not divide by p

Hence $f(x) \Delta g(x)$ is primitive.

Definition 2.19 *Let R be a SP-Ring. Let f(x) in R[x] be a non-zero polynomial in R[x]. Then f(x) is said to be irreducible if f(x) cannot be factored as product of two polynomial in R[x].*

Definition 2.20 *Let R be a SP-Ring. Let F be the set of all elements of the form a/b, where a, b $\in$ R and b $\neq$ 0. F[x] be the set of all polynomial of the form $(a_0/b_0) + (a_1/b_1)x + ...+(a_n/b_n) x^n$, where n is a positive integer and $a_0, a_1,...a_n \in R$.*

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Result 2.21 *(F, ∗, Δ) and (F[x], ∗, Δ) are SP-Rings.*

Theorem 2.22 *Let R be Unique factorization domain.Let F be the set of all quotients of R. Let f(x) ∈ R[x] and deg f(x) > 0. If f(x) is both primitive and irreducible in R[x], then f(x) is also irreducible in F[x].*

Proof:

Let f(x) ∈ R[x] and deg f(x) > 0

Suppose f(x) is irreducible in R[x], but reducible in F[x]

Then f(x) = g(x) Δ h(x), where g(x) and h(x) have the co-efficient in F

By clearing of denomintors and taking out common factor , then f(x) = (a/b) Δ r(x) Δ s(x), where a, b are elements of R and r(x) and s(x) are primitive polynomial in R[x]

b Δ f(x) =b Δ (a/b) Δ r(x) Δ s(x)

⟹ b Δ f(x) =a Δ r(x) Δ s(x)

The content of the left-hand side is b, since f(x) is primitive and both r(x) and s(x) are primitive

By theorem 4.4 r(x) Δ s(x) is primiitive,so the content of the right-hand side is a

Therefore a = b

⟹ f(x) = r(x) Δ s(x), where r(x) and s(x) in R[x], by cancellation law

⟹ f(x) is reducible in R[x] Which is a contradiction to our hypothesis

Hence f(x) is irreducible in F[x].

Theorem 2.23 *Let R be a unique factorization domain. Let F be the set of all quotients of R. Let f(x) ∈ R[x] and deg f(x) > 0. If f(x) is primitive in R[x] and irreducible in F[x], then f(x) is irreducible in R[x].*

Proof:

Suppose f(x) is primitive in R[x] and irreducible in F[x]

To prove f(x) is irreducible in R[x]

Suppose f(x) = g(x) Δ h(x), where g(x), h(x) ∈ R[x]

Since R[x] ⊂ F[x] , g(x), h(x) ∈ F[x]

But f(x) is irreducible in F[x]

Therefore g(x) = k and k ∈ R

⟹ f(x) = k Δ h(x)

But f(x) is primitive

Therefore k must be constant e' in R

Hence f(x) is irreducible in R[x].

Theorem 2.24 *GAUSS LEMMA*

If the primitive polynomial f(x) can be factored as the two polynomials having co-efficient in F, it can be factored as the product of two polynomials having co-efficients in R.

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Proof:

Suppose f(x) = r(x) Δ s(x), where r(x) and s(x) have co-efficient in F

By clearing of denomintors and taking out common factor ,we get

f(x) = (a/b) Δ r(x) Δ s(x), where a, b are elements of R and r(x) and s(x) are primitive polynomial in R[x]

b Δ f(x) =b Δ (a/b) Δ r(x) Δ s(x)

⟹ b Δ f(x) =a Δ r(x) Δ s(x)

The content of the left-hand side is b, since f(x) is primitive and both r(x) and s(x) are primitive, by theorem r(x) Δ s(x) is primitive, so the content of the right-hand side is a

Therefore a = b

⟹ f(x) = r(x) Δ s(x), where r(x) and s(x) in R[x], by cancellation law

Hence f(x) can be factored as the product of two polynomials having co-efficients in R.

Theorem 2.25 *EISENSTEIN CRITERION*

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ *be a primitive polynomial in R[x]. Suppose there exists a prime number p such that* p/a_0 , $p/a_1, \ldots p/a_{n-1}$ *but p does not divide* a_n *and* p^2 *does not divide* a_0*. Then f(x) is irreducible over F.*

Proof:

Given $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ *be a polynomial in R[x]*

To prove that f(x) is irreducible in R[x]

Suppose f(x) is reducible in R[x]

Then $f(x) = (b_0 + b_1 x + \ldots + b_r x^r)\, \Delta\, (c_0 + c_1 x + \ldots + c_s x^s)$*, where* $b_i, b_j \in R$*,* $b_i, b_j \neq 0$*,* $s<n$

Therefore $a_0 = b_0\, \Delta\, c_0$

p^2 *does not divide* a_0 *and* p/a_0

Hence p cannot divide both b_0 *and* c_0

Assume that p diviides c_0 *and p does not divide* b_0

Now $a_n = b_r\, \Delta\, c_s$

Also p does not divide a_n

p does not divide both b_r *and* c_s

Choose m, such that p divides $c_1, c_2, \ldots c_{m-1}$ *and p does not divide* c_m

Clearly $m \leq s < n$

$\Longrightarrow m < n$

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$a_m = b_0 \Delta c_m + b_1 \Delta c_{m-1} + \ldots + b_{m-i} \Delta c_i$, for some i, where $0 \le i < m$

Now p does not divide $b_0 \Delta c_m$

and p divides $(b_1 \Delta c_m + \ldots + b_{m-i} \Delta c_i)$

p does not divide a_m

But p divides a_m since $m < n$, by our hypothesis

Which is a contradiction

Therefore f(x) is irreducible in R[x]

By theorem 4.5

f(x) is also irreducible in F[x].

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