Preface

This volume is the Pre-conference Proceedings of the Second International Conference on Algebra and Discrete Mathematics (ICADM-2020) conducted by the Department of Mathematics, DDE, Madurai Kamaraj University during June 24 -26, 2020 in online mode. The main themes of the conference are Algebra, Discrete Mathematics and their applications. The role of Algebra and Discrete Mathematics in the field of Mathematics has been rapidly increasing over several decades. In recent decades, the graphs constructed out of algebraic structures have been extensively studied by many authors and have become a major field of research. The benefit of studying these graphs is that one may find some algebraic property of the under lying algebraic structure through the graph property and the vice-versa. The tools of each have been used in the other to explore and investigate the problem in deep. This conference is organized with the aim of providing an avenue for discussing recent advancements in these fields and exploring the possibility of effective interactions between these two areas.

The aim of the conference is to introduce research topics in the main streams of Algebra and Discrete Mathematics to young researchers especially research students, and encourage them to collaborate in teams lead by well-known mathematicians from various countries.

This volume makes available a record of articles presented in the conference. This volume contains the papers presented in the conference without any referring process.

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SECURE DOMINATION IN ZERO-DIVISOR GRAPH

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Abstract

Let $\Gamma(G) = \big(V(\Gamma(G)), E(\Gamma(G))\big)$ be a zero divisor graph. A dominating set S of $V(\Gamma(G))$ is a secure dominating set of $\Gamma(G)$ if for every vertex $x \in V(\Gamma(G)) - S$, there exists $y \in N_{\Gamma(Z_n)}(x) \cap S$ such that $(S - \{y\}) \cup \{x\}$ is a domination set. The minimum cardinality of a secure dominating set of $\Gamma(G)$ is called secure domination number. In this paper, the secure domination number of zero divisor graphs is obtained and also studied the structure of this parameter in $\Gamma(Z_n)$.

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I. INTRODUCTION

In 1988, Istvan Beck [2] introduced zero divisor graphs and studied the coloring properties of a graph, whose vertices are all the elements of the ring and two vertices are adjacent if their product is equal to 0. This definition was redefined and simplified in 1999 by Anderson and Livingston [1] to the zero-divisor graphs. The vertices of the zero-divisor graph are all non-zero zero-divisors and two distinct vertices x and y are adjacent if and only if $xy = 0$. On Compared with the Beck's zero divisor graph, Anderson and Livingston [1] excluded zero element, thus the properties of the zero-divisors in the ring were clearly studied. In 1962, Berge [3] defined the concept of the domination number of a graph, calling this as "coefficient of external Stability" and Ore [8] used the name dominating set and domination number for the same concept. In 1977 Cockayne and Hedetniemi [4] made an interesting and extensive survey of the results know at that time about dominating sets in graphs. They have used the notation $\gamma(G)$ for the domination number of a graph, which has become very popular since then. The survey paper of Cockayne and Hedetniemi [4] has generated lot of interest in the study of domination in graphs. The study of domination parameters and related topics is one of most rich and fast developing area in graph theory. There are 2000 research papers were published in different journals. Collection of results and open problems on various dominating sets were published in [6, 7]. Some works on zero divisor graphs and total graphs can be found in [9, 10, 11]. In this paper, secure dominating set of a zero divisor graph is defined and also the secure domination number of $\Gamma(Z_n)$ are obtained under various constraints. Finally some results related to this parameter are stated and proved.

2 Preliminaries

Definition 2.1

Let R be a commutative ring with $1 \neq 0$, and let $Z(R)$ be its set of zero divisors. The zero-divisor graph of R, denoted by $\Gamma(R) = \big(V(\Gamma(R)),\ E(\Gamma(R))\big)$ is the (undirected) graph with vertices $V(\Gamma(R)) = Z(R)^* = Z(R) \setminus \{0\}$, the nonzero zero-divisors of R, and for distinct $x, y \in V(\Gamma(R))$, the vertices x and y are adjacent if and only if $xy = 0$.

$\Gamma(R)$ is the null graph if and only if R is an integral domain.

Note: Here after, we consider a simple graph $\Gamma(R)$ to R with vertices commutative ring R by Z_n and the zero divisor graph $\Gamma(R)$ by $\Gamma(Z_n)$.

Definition 2.2

Let $\Gamma(Z_n)$ be a zero-divisor graph. A subset S of $V(\Gamma(Z_n))$ is a dominating set if every vertex in $V(\Gamma(Z_n)) - S$ is adjacent to atleast one vertex in S.

The minimum of the cardinality of a dominating set is the domination number of a graph $\Gamma(Z_n)$ and its denoted by $\gamma(\Gamma(Z_n))$.

Definition 2.3

Let $\Gamma(Z_n)$ be a simple graph with the vertex set $V(\Gamma(Z_n))$. The neighborhood of x is the set $N_{\Gamma(Z_n)}(x) = \{y \in V(\Gamma(Z_n)) \mid y \odot_n x = 0\}$.

Definition 2.4

A vertex $x \in V(\Gamma(Z_n))$ is said to be a pendant vertex if its neighborhood contains exactly one vertex. i.e., $N_{\Gamma(Z_n)}(x)$ is a singleton set.

Definition 2.5

Let $\Gamma(Z_n)$ be a zero divisor graph. The degree of a vertex y is denoted by $d(y)$ and defined as number of edges incident to it.

The maximum degree of $\Gamma(Z_n)$ is denoted by $\Delta(\Gamma(Z_n)) = \max\{d(y) : y \in V(\Gamma(Z_n))\}$

3 SECURE DOMINATION NUMBER OF $\Gamma(Z_n)$

Definition 3.1

A dominating set S of $\Gamma(Z_n)$ is said to be a secure dominating set if for each vertex $x \in V(\Gamma(Z_n)) - S$, there exist $y \in N_{\Gamma(Z_n)}(x) \cap S$, such that $(S - \{y\}) \cup \{x\}$ is the dominating set.

The minimum cardinality of a minimal secure dominating set is called a secure domination number. It is denoted by the symbol $\gamma^{sd}(\Gamma(Z_n))$, the corresponding minimum secure dominating set is denoted by γ^{sd} −set.

Theorem 3.2

Let $\Gamma(Z_{2p})$ be a zero divisor graph with prime number $p \geq 3$. Then $\gamma^{sd}\left(\Gamma(Z_{2p})\right) = p - 1$.

Proof:

Let $\Gamma(Z_{2p})$ be a zero divisor graph.

Let p be any prime number with $p \geq 3$.

Then $V\left(\Gamma(Z_{2p})\right) = \{2,4,6,..,2(p-1),\ p\}$.

Let $x = 2(p-1)$ and $y = p$ be two vertices, then

$$x \odot_{2p} y = 2(p-1) \odot_{2p} p = 0 \text{ (since } x.y \text{ is a multiple of } 2p)$$

Hence $xy \in E\left(\Gamma(Z_{2p})\right)$

Similarly,

let $x \in V\left(\Gamma(Z_{2p})\right) - \{y\}$ and $y = p$ then $y \odot_{2p} x = 0$.

Consider $S = V\left(\Gamma(Z_{2p})\right) - \{y\}$ is a dominating set and it has $p - 1$ elements

Take $x \in S$, then

For every $y \in V\left(\Gamma(Z_{2p})\right) - S$, there exists $x \in N_{\Gamma(Z_{2p})}(y) \cap S$ such that $\left((S - \{x\}) \cup \{y\}\right)$ is also a dominating set.

Thus S is a γ^{sd} −set.

Hence $\gamma^{sd}\left(\Gamma(Z_{2p})\right) = p - 1$.

Theorem 3.3

Let $\Gamma(Z_{3p})$ be a zero divisor graph with prime number $p > 3$. Then $\gamma^{sd}\left(\Gamma(Z_{3p})\right) = 2$.

Proof:

Let $\Gamma(Z_{3p})$ be a zero divisor graph and $p > 3$ be a prime number.

The vertex set $V\left(\Gamma(Z_{3p})\right) = \{3, 6, 9, \ \dots, 3(p-1),\ p,\ 2p\}$.

Let $y \in \Gamma(Z_{3p})$ with $d(y) = \Delta\left(\Gamma(Z_{3p})\right)$.

Let z be a another vertex with $d(z) = \Delta$ in $\Gamma(Z_{3p})$ then

Either $z = p, y = 2p$ (or) $z = 2p, y = p$

Then $z \odot_{3p} y = 2p \odot_{3p} p \neq 0$ (since $z.y$ is not divisible by $3p$).

$$zy \notin E\left(\Gamma(Z_{3p})\right)$$

Let x be any vertex in $V\left(\Gamma(Z_{3p})\right) - \{z, y\}$ such that $z \odot_{3p} x = y \odot_{3p} x = 0$.

Then $zx, yx \in E\left(\Gamma(Z_{3p})\right)$.

Let $S = \{z, y\} \subseteq V\left(\Gamma(Z_{3p})\right)$ is a dominating set.

For every $x \in V\left(\Gamma(Z_{3p})\right) - S$, there exists $z \in N_{\Gamma(Z_{3p})}(x) \cap S$ such that $(S - \{z\}) \cup \{x\} = \{y, x\}$ is a also dominating set. Thus S is a γ^{sd} –set.

Hence $\gamma^{sd}\left(\Gamma(Z_{3p})\right) = 2$.

Theorem 3.4

Let $\Gamma(Z_{4p})$ be a zero divisor graph with prime number $p \geq 5$. Then $\gamma^{sd}(\Gamma(Z_{4p})) = 3$.

Proof:

Let $\Gamma(Z_{4p})$ be a zero divisor graph and $p \geq 5$ be a prime number.

The vertex set of $V\left(\Gamma(Z_{4p})\right) = \{2, 4, 6, \ . \ . \ . \ , \ 2(2p-1), \ p, \ 2p, \ 3p\}$.

Let $x = 2p$ and t is any even number from 2 to $2(2p-1)$.

Clearly $x \odot_{4p} t = 2p \odot_{4p} 2(2p-1) = 0$ (since $x.t$ is a multiple of $4p$)

Hence $xt \in E\left(\Gamma(Z_{4p})\right)$

Also, let $x = 2p$, $y = p$ and $z = 3p$ are in $V\left(\Gamma(Z_{4p})\right)$.

Then $x \odot_{4p} y = 2p \odot_{4p} p \neq 0$, (since $x.y$ is not divisible by $4p$).

$y \odot_{4p} z = p \odot_{4p} 3p \neq 0$ and , (since $y.z$ is not divisible by $4p$).

$x \odot_{4p} z = 2p \odot_{4p} 3p \neq 0$ (since $x.z$ is not divisible by $4p$).

$$xy, yz, zx \notin E\left(\Gamma(Z_{4p})\right)$$

Let $u_n = 4n \in V\left(\Gamma(Z_{4p})\right)$ where $n = 1,2,3, \dots, p-1$.

Then $u_n \odot_{4p} y = u_n \odot_{4p} z = 0$ where $n = 1,2,3, \dots, p-1$.

Hence $u_n y, u_n z \in E\left(\Gamma(Z_{4p})\right)$.

Choose $u_n^* \in \{u_n\}$ and let $S = \{x, y, u_n^*\} \subseteq V\left(\Gamma(Z_{4p})\right)$ be a dominating set.

For every $u_n^{**} \in V\left(\Gamma(Z_{4p})\right) - S$, there exists $y \in N_{\Gamma(Z_{4p})}(u_n^{**}) \cap S$ such that $(S - \{y\}) \cup \{u_n^{**}\} = \{x, u_n^*, u_n^{**}\}$ is also a dominating set, for some $u_n^*, u_n^{**} \in \{u_n\}$ and $u_n^* \neq u_n^{**}$.

Thus S is a γ^{sd} –set.

Hence $\gamma^{sd}(\Gamma(Z_{4p})) = 3$.

Theorem 3.5

Let $\Gamma(Z_{5p})$ be a zero divisor graph with prime number $p > 5$. Then $\gamma^{sd}(\Gamma(Z_{5p})) = 3$.

Proof:

Let $\Gamma(Z_{5p})$ be a zero divisor graph with $p > 5$.
The vertex set of $V(\Gamma(Z_{5p})) = \{5,10,15,. \ldots,5(p-1), p, 2p, 3p, 4p\}$.
Now, the vertex set $V(\Gamma(Z_{5p}))$ can be partition in the two parts V_1 and V_2 with $V_1 \cup V_2 = V(\Gamma(Z_{5p}))$.
Consider $V_1 = \{5,10,15,. \ldots,5(p-1)\}$ and $V_2 = \{p, 2p, 3p, 4p\}$.
Let $x, y \in V_1$
Then $x \odot_{5p} y \neq 0$, (since $x.y$ is not divisible by $5p$).
Similarly,
Let $z, w \in V_2$.
Then $z \odot_{5p} w \neq 0$, (since $z.w$ is not divisible by $5p$).
Therefore $xy, zw \notin E\left(\Gamma(Z_{5p})\right)$
Hence no vertices of V_1 is adjacent to any vertices of V_1.
Similarly, no vertices of V_2 is adjacent to any vertices of V_2.
Let $x \in V_1$ and $y \in V_2$.
Then $x \odot_{5p} y = 0$. (since $x.z$ is a multiple of $5p$)
Hence $xy \in E\left(\Gamma(Z_{5p})\right)$.
Therefore, every vertex in V_1 is adjacent to any vertices in V_2.
Consider $S = \{x, z, w\} \subseteq V(\Gamma(Z_{5p}))$ is a dominating set.
For every $y \in V(\Gamma(Z_{5p})) - S$, there exists $z \in N_{Z_{5p}}(y) \cap S$ such that $(S - \{z\}) \cup \{y\} = \{x, w, y\}$ is also a dominating set.
Thus S is a γ^{sd} –set.
Hence $\gamma^{sd}(\Gamma(Z_{5p})) = 3$.

Theorem 3.6

Let $\Gamma(Z_{7p})$ be a zero divisor graph with prime number $p > 7$, Then $\gamma^{sd}\left(\Gamma(Z_{7p})\right) = 3$.

Proof:

Let $\Gamma(Z_{7p})$ be a zero divisor graph and $p > 7$ be a prime number.
The vertex set of $\Gamma(Z_{7p}) = \{7,14,21,..,7(p-1), p, 2p, 3p, 4p, 5p, 6p\}$.
Clearly , the vertex V canbe partition in the two parts V_1 & V_2.
$V_1 = \{7,14,21,...,7(p-1)\}$ and $V_{2=}\{p, 2p, 3p, 4p, 5p, 6p\}$.
Let $x \in V_1$ & $z \in V_2$
Then $x \odot_{7p} z = 0$ (since $x.z$ is a multiple of $7p$)
Hence $xz \in E\left(\Gamma(Z_{7p})\right)$.
Let $x, y \in V_1$
Then $x \odot_{7p} y \neq 0$ (since $x.y$ is not divisible by $7p$)
Similarly,
Let $z, w \in V_2$

.

Then $z \odot_{7p} w \neq 0$ (since $z.w$ is not divisible by $7p$)

$$xy, zw \notin E\left(\Gamma(Z_{7p})\right)$$

Therefore, every vertex in V_1 is adjacent to any vertex in V_2 and vice versa.

Let $S = \{x, y, z\}$ where $x, y \in V_1$ & $z \in V_2$. For every $w \in V\left(\Gamma(Z_{7p})\right) - S$, there exist $x \in N_{Z_{7p}}(w) \cap S$ such that $S - \{x\} \cup \{w\} = \{y, z, w\}$ is a dominating set.

Thus S is a γ^{sd} −set.

Hence $\gamma^{sd}\left(\Gamma(Z_{7p})\right) = 3$.

Theorem 3.7

Let $\Gamma(Z_{2p})$ be a zero where $p \geq 3$, with p vertices and maximum vertex degree $\Delta(\Gamma(Z_{2p}))$ then $\gamma_s\left(\Gamma(Z_{2p})\right) = 2p - 2 - \Delta\left(\Gamma(Z_{2p})\right)$ if and only if $\Gamma(Z_{2p})$ is a star graph.

Proof:

Let y be a vertex with maximum degree $\Delta(\Gamma(Z_{2p}))$.

If $\Gamma(Z_{2p})$ is a star, with y is the root, then $\Gamma(Z_{2p})$ has exactly $\Delta(\Gamma(Z_{2p}))$ branches from y.

Then the number of leaves (edges) in $\Gamma(Z_{2p})$ is exactly $\Delta\Gamma(Z_{2p})$.

Since $\gamma^{sd}\left(\Gamma(Z_{2p})\right) = p - 1$.

That is, $p - 1 = 2p - 2 - \Delta\left(\Gamma(Z_{2p})\right) = p - 1$.

Therefore $\gamma_s\left(\Gamma(Z_{2p})\right) = 2p - \Delta\left(\Gamma(Z_{2p})\right) - 2$.

Conversely, if $\Gamma(Z_{2p})$ is not a star, then there exist a vertex other than y with degree not less than 3 in $\Gamma(Z_{2p})$.

This shows that $\Gamma(Z_{2p})$ has more than $\Delta\left(\Gamma(Z_{2p})\right)$ edges, which is contradiction.

Hence the theorem.

REFERENCES

[1] D.F. Anderson and P.S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra, 217 (1999) 434-447.

[2] I.Beck, Colouring of commutative ring, J. Algebra, 159(1993), 500-514.

[3] C. Berge, Theory of Graphs and its Applications, Methuen, London 1962.

[4] E.J. Cockayne, S.T. Hedetniemi, Towards a Theory of Domination in Graphs. Networks, 7, (1977), 247-261.

[5] E.J. Cockayne, O. Favaron and C.M. Mynhardt, Secure domination, weak Roman domination and forbidden subgraphs, Bull. Inst. Combin. Appl., 39(2003), 87-100.

[6] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), Fundamentals of Domination in graphs, Marcel Dekker, Inc. New York, 1998.

[7] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), Domination in graphs – advanced topics, Marcel Dekker, Inc. New York, 1998.

[8] O.Ore, Thoery of Graphs, Amer. Math. Soc. Transl., 38(1962), pp. 206-212.

[9] J. Ravi Sankar, S. Meena, *Connected Domination number of a Commutative Ring*, International Journals of Mathematical Research, Vol. 5, 2013, 5-11.

[10] N.H. Shukar, H.Q. Mohamed and A.M. Ali, The zero-divisor graph of $Z_{\{p^n q\}}$, Int. Journal of Algebra, Vol. 6, 2012, No-22, 1049-1055.

[11] T. Tamizh Chelvam and T. Asir, Domination in the total graph of a commutative ring, Journal of Combinatorial Mathematics and Combinatorial Computing, 87(2013), 147-158.