Preface

This volume is the Pre-conference Proceedings of the Second International Conference on Algebra and Discrete Mathematics (ICADM-2020) conducted by the Department of Mathematics, DDE, Madurai Kamaraj University during June 24 -26, 2020 in online mode. The main themes of the conference are Algebra, Discrete Mathematics and their applications. The role of Algebra and Discrete Mathematics in the field of Mathematics has been rapidly increasing over several decades. In recent decades, the graphs constructed out of algebraic structures have been extensively studied by many authors and have become a major field of research. The benefit of studying these graphs is that one may find some algebraic property of the under lying algebraic structure through the graph property and the vice-versa. The tools of each have been used in the other to explore and investigate the problem in deep. This conference is organized with the aim of providing an avenue for discussing recent advancements in these fields and exploring the possibility of effective interactions between these two areas.

The aim of the conference is to introduce research topics in the main streams of Algebra and Discrete Mathematics to young researchers especially research students, and encourage them to collaborate in teams lead by well-known mathematicians from various countries.

This volume makes available a record of articles presented in the conference. This volume contains the papers presented in the conference without any referring process.

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International Journal of Computer Science ISSN: 2348-6600

Reference ID: IJCS-364 Volume 8, Issue 2, No 1, 2020. Page No : 2459-2465

International Conference on Algebra and Discrete Mathematics

ID:20ADM67 June 24-26,2020

*δ- Topological Vector Spaces

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Abstract

The aim of this paper is to introduce the class of *δ-topological vector spaces as a structure in topology, in which a vector space X over a topological field K ($\mathbb{R}$ or $\mathbb{C}$) is endowed with a topology τ such that the vector space operations are *δ-continuous with respect to τ. It is characterized in terms of *δ-open sets. Further more, it has been investigated the condition for a subspace has to be a *δ-open sets in *δ-topological vector spaces.

Keywords: *δ-open set, *δ-continuous, *δ-topological vector spaces.

AMS Subject Classification: 54C08, 57N17, 57N99.

1. Introduction

In 2015, Khan et al [3] introduced and studied the s-topological vector spaces which are a generalization of topological vector spaces. In 2016, Khan and Iqbal [4] introduced the irresolute topological vector spaces which are independent of topological vector spaces. In 2019, β-topological vector spaces have been introduced by S. Sharma and M. Ram [9]. After that S, Sharma et al [10] defined and investigated almost β-topological vector spaces.

The aim of this paper is to introduce the class of *δ-topological vector spaces as a structure in topology, in which a vector space X over a topological field K ($\mathbb{R}$ or $\mathbb{C}$) is endowed with a topology τ such that the vector space operations are *δ-continuous with respect to τ. It is characterized in terms of *δ-open sets. Further more, it has been investigated the condition for a subspace has to be a *δ-open sets in *δ-topological vector spaces.

Throughout the present paper (X,τ) (Simply X) always mean topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space $X, cl(A)$ and $int(A)$ denote the closure and the interior of A respectively.

2. Preliminaries

Definition 2.1.[6] A subset A of a topological space (X,τ) is called

(i) generalized closed (briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

(ii) generalized open (briefly g-open) if $X\backslash A$ is closed in X.

Definition 2.2.[1] Let A be a subset of X. Then

(i) generalized closure of A is defined the intersection of all g-closed sets containing A, and is denoted by $cl^*(A)$.

(ii) generalized interior of A id defined as the union of all g-open subsets of A and is denoted by $int^*(A)$.

Definition 2.3.[8] A subset A of a topological space (X, τ) is

(i) $Regular^* - open$ if $A = int(cl^*(A))$.

(ii) $Regular^* - closed$ if $A = cl(int^*(A))$.

The $Regular^* interior$ of A is defined as the union of all $Regular^* - open$ subsets of A and is denoted by $r^* int(A)$.

Definition 2.4.[7] The *δ- interior of a subset A of X is the union of all $Regular^* - open$ sets of X contained in A and is denoted by $int_{*_\delta}(A)$.

A subset A of a topological space (X, τ) is called *δ–open if $A = int_{*_\delta}(A)$ ie, a set is *δ–open if it is the union of $Regular^* - open$ sets. The complement of a *δ–open set is called *δ–closed set in X.

Definition 2.5.[7] A function $f: (X, \tau) \to (Y, \sigma)$ is said to be *δ-continuous if $f^{-1}(V)$ is *δ-closed in (X, τ) for every closed set in (Y, σ).

Definition 2.6.[2] A topological vector space is a real (or complex) vector space over a field F with topology τ such that the addition mapping $m: X \times X \to X$ defined by $m((x,y)) = x + y$, and the scalar multiplication $M: F \times X \to X$ defined by $M((\lambda, x)) = \lambda x$ are continuous for each λ in F and x, y in X.

Definition 2.7.[3] If X is a vector space, e denotes its identity element, and for a fixed $x \in X$, $xT: X \to X; y \to x + y$ and $T_x: X \to X;\ y \to y + x$, denote the left and right translation by A respectively.

Definition 2.8. Let S be a subset of a vector space. If S is said to be symmetric then it is symmetric with respect to the addition group structure of the vector space. ie if $S = -S = \{-x : x \in S\}$. For any subset S of a vector space, $S \cup (-S)$ and $S \cap (-S)$ are symmetric sets.

3. *δ-Topological Vector Spaces

In this section, we are defining the *δ-topological vector space over a real or complex field and derives its characterization.

Definition 3.1. A *δ-topological vector space is a vector space X over a field F (real or complex) with a topology τ with the following conditions.

(i) The vector addition mapping $m: X \times X \to X$ define by $m\big((x,y)\big) = x + y$, for each x, y in X is *δ-continuous

(ii) The scalar multiplication mapping $M: F \times X \to X$ which define by $M((\lambda, x)) = \lambda x$ for each λ in F and x, y in X is *δ-continuous.

Then the pair $(X_{(F)}, \tau)$ is called Topological vector space. In short, it is denoted by X, a *δ-TVS.

Theorem 3.2. A vector space X over a field F with a topology τ is *δ-topological vector space iff

(i) for each pair x, y in X and for each open set W of $x + y$ in X, there exists a *δ-open set U in X containing x and a *δ-open set V in X containing y such that $U + V \subseteq W$.

(ii) for each x in F, x in X and for each open set W containing the point αx, there exists a *δ-open sets U containing α in F and V containing x in X such that $U.V \subseteq W$.

Proof: (i) Assume that $(X_{(F)}, \tau)$ is a *δ-topological vector space. Let x and y be any two points in X and W be any open set containing $x + y$ in X. Since vector addition map $m: X \times X \to X$ is *δ-continuous, there exists a *δ-open set $P \times Q$ containing (x, y) such that $f(P \times Q) \subseteq W$. Since P and Q are *δ-open sets containing x and y respectively, there exists *δ-open sets U containing x and V containing y such that $x \in U \subseteq P$ and $y \in V \subseteq Q$. Then $(x, y) \in U \times V \subseteq P \times Q$. Hence $U + V = f(U \times V) \subseteq f(P \times Q) \subseteq W$.

(ii) Let $\alpha \in F$, $x \in X$ and W be any open set containing αx. Scalar multiplication map $M: F \times X \to X$ is *δ-continuous, there exists a *δ-open set $U \times V$ in $F \times X$ containing (α, x) such that $M(U \times V) \subseteq W$. Since U is *δ-open set in F containing α, there exists a *δ-open set U_1 in K containing α such that $\alpha \in U_1 \subseteq U$. Similarly, there exists a *δ-open set V_1 in X containing x such that $x \in V_1 \subseteq V$. Now $(\alpha, x) \in U_1 \times V_1 \subseteq U \times V$. Then $M((\alpha, x)) \in M(U \times V) \subseteq W$. Hence $\alpha x \in M(U \times V) = UV \subseteq W$.

Conversely, assume that the given two conditions hold. Let $(x, y) \in X \times X$ be any point and W be any open set in X containing the point $f\big((x, y)\big) = x + y$. By hypothesis, there exists a *δ-open sets U containing x and V containing y such that $U+V\subseteq W$. Then $U\times V$ is a *δ-open set in $X\times X$ containing the point (x,y) such that $f(U\times V)=U+V\subseteq W$. Hence the vector addition mapping m is *δ-continuous. Now let $\alpha\in F, x\in X$ be any points and W be any open set in X containing $M\big((\alpha,x)\big)=\alpha x$. By hypothesis, there exists a *δ-open sets U in F containing α and V in X containing x such that $UV\subseteq W$. Since U is a *δ-open set in F containing α, there exists a *δ-open set U_1 in F containing α such that $\alpha\in U_1\subseteq U$. Similarly, there exists a *δ-open set V_1 in X containing x such that $x\in V_1\subseteq V$. Now $(\alpha,x)\in U_1\times V_1\subseteq U\times V$. Then $M\big((\alpha,x)\big)\in M(U\times V)=UV\subseteq W$. This implies that $U\times V$ is a *δ-open set in $F\times X$ containing the point (α,x) such that $M(U\times V)\subseteq W$. Hence the scalar multiplication mapping M is *δ-continuous. Therefore $(X_{(F)},\tau)$ is a *δ-topological vector space.

Theorem 3.3. Let $(X_{(F)},\tau)$ be a *δ-topological vector space. If A is open in $(X_{(F)},\tau)$, then the following are true.

(i) $x+A$ is a *δ-open for each $x\in X$.

(ii) αA is a *δ-open for each non-zero scalar α in F.

Proof:(i) Let $y\in x+A$, then there exists *δ-open sets U and V containing $-x$ and y respectively such that $U+V\subseteq A$. Then $-x+V\subseteq U+V\subseteq A$. This implies that $V\subseteq x+A \Longrightarrow y\in int_{*_\delta}(x+A)$. Therefore $x+A\subseteq int_{*_\delta}(x+A)$. Always $int_{*_\delta}(x+A)\subseteq x+A$. Hence $x+A=int_{*_\delta}(x+A)$. Thus $x+A$ is a *δ-open set in $(X_{(F)},\tau)$.

(ii) Let $x\in\alpha A$. Then there exists *δ-open sets U in F containing $1/\alpha$ and Vin X containing x such that $UV\subseteq A$. This implies that $x\in V\subseteq \alpha A$. Then $x\in int_{*_\delta}(\alpha A)$. Thus $\alpha A\subseteq int_{*_\delta}(\alpha A)$. Always $int_{*_\delta}(\alpha A)\subseteq\alpha A$. Hence $\alpha A=int_{*_\delta}(\alpha A)$. Thus αA is a *δ-open set in $(X_{(F)},\tau)$.

Theorem 3.4. Let $(X_{(F)},\tau)$ be a *δ-topological vector space. If A is open subset of X, then $A+B$ is a *δ-open set in X for any subset B of X.

Proof: Given A is any open subset of X. Let B be any open subset of X. Then by Theorem 3.3, $y+A$ is a *δ-open set in X for all $y\in B$. Now $A+B=\bigcup_{y\in B}(A+y)$. Since arbitrary union of *δ-open set is *δ-open, $A + B = \cup_{y \in B}(A + y)$ is *δ-open in X. That is, $A + B$ is *δ-open in X.

Theorem 3.5. Every open subspace of an *δ-topological vector space is also *δ-topological vector space.

Proof: Let $(X_{(F)}, \tau)$ be any *δ-topological vector space and Y be any subspace of X. Let x, y be any two distinct points of Y. Let W be any open set containing the point $x + y$ in the subspace Y. By definition of *δ-topological vector space, there exists a *δ-open set U of x and a *δ-open set V of y in X such that $U + V \subseteq W$. By the definition of subspace topology, the sets $A = U \cap Y$ and $B = V \cap Y$ are *δ-open sets in Y containing x and y respectively. Also $A + B = (U \cap Y) + (V \cap Y) = (U + V) \cap Y \subseteq U + V \subseteq W$. Suppose that $\alpha \in F, x \in Y$ and let W be any open set in Y containing αx in Y. Since W is open in Y and Y is open in X, W is open in X containing the point αx. By hypothesis, there exists an *δ-open set $U \subseteq F$ of α and $V \subseteq X$ of y such that $UV \subseteq W$. By the definition of subspace topology, the set $A = U \cap F$ is *δ-open set in F containing the point α and the set $B = V \cap F$ is *δ-open set in Y containing the point y. Also $AB \subseteq UV \subseteq W$. Hence every open subspace of an *δ-topological vector space is also *δ-topological vector space.

Theorem 3.6. Let $(X_{(F)}, \tau)$ be a *δ-topological vector space and $Y \subseteq X$; (Y_F, τ_Y) be a subspace of $(X_{(F)}, \tau)$. Then Y is *δ-open in X provided that it contains a non-empty *δ-open subset of X.

Proof: Let U be any non-empty *δ-open subset of X such that $U \subseteq Y$. Since any *δ-open set is open, U is open in X. Let $y \in Y$ be arbitrary, then $y + U$ is *δ-open subset of X such that $y + U \subseteq Y$. Then $y \in int_{*_\delta}(Y)$. This implies that $Y \subseteq int_{*_\delta}(Y)$. Always $int_{*_\delta}(Y) \subseteq Y$. Hence $Y = int_{*_\delta}(Y)$. Thus Y is *δ-open subset of X.

Theorem 3.7. In a *δ-topological vector space $(X_{(F)}, \tau)$, for any *δ-open set U containing O, there exists a symmetric *δ-open set V containing O such that $V + V \subseteq U$.

Proof: Let U be any *δ-open set containing O. Since every *δ-open set is open, U is a open set in X containing $O = O + O$. By the definition of *δ-TVS, there exists *δ-open sets V_1 and V_2 containing O such that $V_1 + V_2 \subseteq U$. By the definition symmetric, $V_1 \cap (-V_1)$ and $V_2 \cap (-V_2)$ are symmetric sets containing O. Since intersection of two symmetric sets is always symmetric, $[V_1 \cap (-V_1)] \cap [V_2 \cap (-V_2)] = V$ is a symmetric set containing O. Clearly $V = [V_1 \cap (-V_1)] \cap [V_2 \cap (-V_2)] \subseteq V_1$ and $V = [V_1 \cap (-V_1)] \cap [V_2 \cap (-V_2)] \subseteq V_2$. Then $V + V \subseteq V_1 + V_2 \subseteq U$.

Theorem 3.8. In a *δ-topological vector space $(X_{(F)}, \tau)$, a scalar multiple of a *δ-closed set is *δ-closed for any $\alpha \in F$.

Proof: Let U be any *δ-closed subset of X and $\alpha \in F$ be arbitrary. $(\alpha U)^c = X \backslash \alpha U = \alpha(X \backslash U) = \alpha U^c$. Since U is *δ-closed subset of X, U^c is *δ-open subset of X. Since every *δ-open set is open, U^c is an open subset of X. By Theorem 3.3, αU^c is a *δ-open subset of X. Then $(\alpha U)^c$ is a *δ-open. So αU is *δ-closed subset of X.

Theorem 3.9. Let A be any closed subset of a *δ-topological vector space $(X_{(F)}, \tau)$. Then the following are true.

(i) $x + A$ is *δ-closed for each $x \in X$.

(ii) αA is a *δ-closed for each non-zero scalar α in F.

Proof:(i) Let $y \in cl_{*\delta}(x + A)$. Now consider $Z = -x + y$ and let W be any open set in X containing Z. Then by definition of *δ-topological vector space, there exists *δ-open sets U and V in X such that $-x \in U$, $y \in V$ and $U + V \subseteq W$. Since $y \in cl_{*\delta}(x + A)$, $(x + A) \cap Y \neq \emptyset$. Then there is $a \in (x + A) \cap V$. Now $-x + a \in A \cap (U + V) \subseteq A \cap W \Rightarrow A \cap W \neq \emptyset \Rightarrow Z \in cl(A) = A \Rightarrow y \in x + A$. Hence $cl_{*\delta}(x + A) \subseteq x + A$. Always $x + A \subseteq cl_{*\delta}(x + A)$. Thus $x + A = cl_{*\delta}(x + A)$. This proves that $x + A$ is *δ-closed in X.

(ii) Assume that $x \in cl_{*\delta}(\alpha A)$ and let W be any open neighborhood of $Y = \frac{1}{\alpha}x$ in X. Since $(X_{(F)}, \tau)$ is *δ-TVS, there exists *δ-open sets U in F containing $\frac{1}{\alpha}$ and V in X containing x such that $UV \subseteq W$. By hypothesis, $(\alpha A) \cap V \neq \emptyset$. Therefore there is $a \in (\alpha A) \cap V$. Now $\frac{1}{\alpha}a \in A \cap (UV) \subseteq A \cap W \Rightarrow A \cap w \neq \emptyset \Rightarrow y \in cl(A) = A \Rightarrow x \in \alpha A$.

Then $cl_{*\delta}(\alpha A) \subseteq \alpha A$. Always $\alpha A \subseteq cl_{*\delta}(\alpha A)$. Hence $\alpha A = cl_{*\delta}(\alpha A)$. Thus αA is $*\delta -$ closed set in X.

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