Preface

This volume is the Pre-conference Proceedings of the Second International Conference on Algebra and Discrete Mathematics (ICADM-2020) conducted by the Department of Mathematics, DDE, Madurai Kamaraj University during June 24 -26, 2020 in online mode. The main themes of the conference are Algebra, Discrete Mathematics and their applications. The role of Algebra and Discrete Mathematics in the field of Mathematics has been rapidly increasing over several decades. In recent decades, the graphs constructed out of algebraic structures have been extensively studied by many authors and have become a major field of research. The benefit of studying these graphs is that one may find some algebraic property of the under lying algebraic structure through the graph property and the vice-versa. The tools of each have been used in the other to explore and investigate the problem in deep. This conference is organized with the aim of providing an avenue for discussing recent advancements in these fields and exploring the possibility of effective interactions between these two areas.

The aim of the conference is to introduce research topics in the main streams of Algebra and Discrete Mathematics to young researchers especially research students, and encourage them to collaborate in teams lead by well-known mathematicians from various countries.

This volume makes available a record of articles presented in the conference. This volume contains the papers presented in the conference without any referring process.

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SOME PROPERTIES OF THE SQUARE GRAPH OF FINITE ABELIAN GROUPS

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ABSTRACT. Let G be a finite abelian group. The square graph of G is the simple undirected graph with vertex set G in which two distinct vertices x and y are adjacent if and only if $x+y=2t$ for some $2t \neq 0$ and $t \in G$ where 0 is the identity of the group. In this paper, we discuss the diameter and the girth of the graph $\Gamma_{sq}(G)$. Using these, we obtain the independence number and the clique number of $\Gamma_{sq}(G)$.

1. INTRODUCTION

Throughout this paper R denoted a commutative ring with identity and $R^* = R \setminus \{0\}$. The definition of Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups which are described by a set of generators. There are several other graph constructions from finite groups and rings [1, 2]. The set $Sq(R)$ of squares of R (elements of form x^2 for some $x \in R$) is a very interesting subset from algebraic point of view. Sen Gupta and Sen [8] introduced the square element graph of a finite commutative ring and studied its properties. The square element graph $Sq(R)$ over R is the simple undirected graph with vertex set $V = R^*$ and two vertices a, b are adjacent if and only if $a + b = x^2$ for some $x \in R^*$. Sen Gupta and Sen [9] further generalized the square element graph $Sq(R)$ by defining it over any ring R with unity. Snowden [10] studied this graph for finite full transformation semigroups. Let G be a finite abelian group.The square element graph over G, $\Gamma_{sq}(G)$, is the simple undirected graph with vertex set G in which two distinct vertices x and y are adjacent if and only if $x + y = 2t$ for some $2t \neq 0$ and $t \in G$ where 0 is the identity of the group. In this paper, we concentrate on the square element graph $\Gamma_{sq}(G)$ of a finite group G. In section 2 of this paper, we obtain the diameter and the girth of $\Gamma_{sq}(G)$. We give a condition for $\Gamma_{sq}(G)$ to be self-centered. Also, we obtain the independence number, the clique number and the chromatic number of $\Gamma_{sq}(G)$.

2. PRELIMINARIES

First let us recollect some basic definitions of graph theory which are essential for this paper. By a graph $\Gamma = (V, E)$, we mean Γ is a finite graph with vertex set V and edge set E. A graph Γ is said to be *complete* if each pair of distinct vertices is joined by an edge. We use K_n to denote the complete graph with n vertices. A graph Γ is said to be *connected* if every distinct pair of vertices in Γ has a path. For a vertex $v \in V(G)$, $N(x)$ is the set of all vertices in G which are adjacent to v and $N[x] = N(x) \cup \{x\}$. The distance $d(u, v)$ between the vertices u and v in

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Γ is the length of the shortest path between u and v. If no path exists between u and v in Γ, then $d(a,b)=\infty$. For a vertex $v \in V(\Gamma)$, the *eccentricity* of v is the maximum distance from v to any vertex in Γ. That is, $e(v)=\max\{d(v,w) : w \in V(\Gamma)\}$. The *radius* of Γ is the minimum eccentricity among the vertices of Γ. i.e., $\text{radius}(\Gamma)=\min\{e(v) : v \in V(G)\}$. The *diameter* of Γ is the maximum eccentricity among the vertices of Γ. i.e., $\text{diameter}(\Gamma)=\max\{e(v) : v \in V(G)\}$. The *girth* of Γ is the length of a shortest cycle in Γ and is denoted by $gr(\Gamma)$. The *degree* of a vertex v is the number of the edges in Γ which are incident with v. A *clique* of Γ is a maximal complete subgraph of Γ and the number of vertices in the largest clique of Γ is called the *clique number* of Γ and is denoted by $\omega(\Gamma)$. An *independent set* is a set of vertices in a graph, in which no two vertices are adjacent and cardinality of maximal independent set is called the *independent number*. [11, 12].

Let G be a finite abelian group and $|G| = 2^{\alpha} \times p_1^{\alpha_1} \times \ldots \times p_r^{\alpha_r}$ where 2 and $p_i's$ are distinct primes and $\alpha, \alpha_i \in \mathbb{Z}^+ \cup \{0\}$ for $1 \leq i \leq r$. Then G is the direct product of finite cyclic groups. i.e., $G \cong \mathbb{Z}_{2^{\alpha}} \times \prod_{i=1}^{r} \mathbb{Z}_{p_i^{\alpha_i}}$. If $|G|$ is divisible by 2 and $\alpha = m_1+\cdots+m_k$, for $\alpha, m_1, \ldots m_k \in \mathbb{Z}^+$, then we have $G \cong \prod_{i=1}^{k} \mathbb{Z}_{2^{m_i}} \times \mathbb{Z}p_1^{\alpha_1} \times \ldots \times \mathbb{Z}p_r^{\alpha_r}$.

Remark 2.1. For a finite abelian group G, let $Sq(G) = \{2t \mid t \in G\} \subseteq G$ and $Od(G) = G \setminus Sq(G)$. Note that two distinct vertices x and y are adjacent in $\Gamma_{sq}(G)$ if and only if $x+y \in Sq(G) \setminus \{0\}$.

Remark 2.2. (1). If $|G|$ is not divisible by 2, then $G = Sq(G)$ and $Od(G) = \phi$.

(2). Assume that $|G|$ is divisible by 2 and so $\alpha \geq 1$. Consider a partition $P(\alpha) = m_1 + \cdots + m_k$ of α. Here $k \geq 1$. Now, we respect to a partition of α, one can associate 2^k, k-tuples of 0's and 1's. For a k-tuple, $\ell = (a_1, a_2, \ldots, a_k)$, let

$$X_\ell = \prod_{i=1}^{k} H_i \times \mathbb{Z}p_1^{\alpha_1} \times \ldots \times \mathbb{Z}p_r^{\alpha_r}. \text{ where } H_i = \begin{cases} Sq(\mathbb{Z}_{2^{m_i}}) & \text{if } a_i = 0; \\ Od(\mathbb{Z}_{2^{m_i}}) & \text{if } a_i = 1. \end{cases}$$

(3). One can check that in either case, $Sq(G) = X_{(0,0,\ldots,0)} = X_1$, $Od(G) = \bigcup_{i=2}^{2^k} X_i$ and $|X_\ell| = \frac{|G|}{2^k}$, For $\ell,\ 1 \leq \ell \leq 2^k$.

3. Basic properties of $\Gamma_{sq}(G)$

In this section, we list out some basic properties of the square graph $\Gamma_{sq}(G)$ of finite abelian group G. In the following theorem, we show that $\Gamma_{sq}(G)$ is connected and obtain diameter and girth of the graph $\Gamma_{sq}(G)$.

Theorem 3.1. [3] *Let G be a finite abelian group. Then the following are equivalent:*

(1) *$\Gamma_{sq}(G)$ is connected.*
(2) *All elements of G are squares.*
(3) *$|G|$ is odd.*

In this following, we obtain the diameter and girth of $\Gamma_{sq}(G)$ where $|G|$ is not divisible by 2.

Theorem 3.2. *Let G be a finite abelian group and $|G|$ is not divisible by 2. Then*

(1) $diam(\Gamma_{sq}(G)) = 2$

(2) $gr(\Gamma_{sq}(G)) = 3$.

Proof. Assume that $|G|$ is not divisible by 2. As observed in Remark 2.2(1), we have $G = Sq(G)$.

(1). If $0 \neq x \in G$ is an element in G. Then there exists an element $y \in G$ such that $x \neq y$ and $x + y = 0$. Thus x is not adjacent to y but there exists a path $x - 0 - y$ of length two in G. Thus $\mathrm{diam}(\Gamma_{sq}(G)) = 2$.

(2). If $|G|$ is not divisible by 2, then $|G| \geq 3$. Let $0, x, y \in G$ with $x + y \neq 0$. Then $0 - x - y - 0$ is a cycle of length 3 in G and so $\mathrm{gr}(\Gamma_{apq}(G)) = 3$. □

Lemma 3.3. [7] *Let G be a finite abelian group, $|G| = 2^{\alpha} \times p_1^{\alpha_1} \times \ldots \times p_r^{\alpha_r}$ where 2 and $p_i's$ are distinct primes and $\alpha, \alpha_i \in \mathbb{Z}^+ \cup \{0\}$ for $1 \leq i \leq r$.. If $x \in X_i$ and $y \in X_j$ for $i \neq j$, then x is not adjacent to y in $\Gamma_{sq}(G)$.*

In the following theorem, we obtain the independence number of $\Gamma_{sq}(G)$.

Theorem 3.4. *Let G be a finite abelian group, $|G| = 2^{\alpha} \times p_1^{\alpha_1} \times \ldots \times p_r^{\alpha_r}$ where 2 and $p_i's$ are distinct primes and $\alpha, \alpha_i \in \mathbb{Z}^+ \cup \{0\}$ for $1 \leq i \leq r$ and $G \cong \prod_{i=1}^{k} \mathbb{Z}_{2^{m_i}} \times \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_r}$, $\alpha \in \mathbb{Z}^+$ and $\sum_{i=1}^{k} m_i = \alpha$. Then the independence number*

$$\beta(\Gamma_{sq}(G)) = \begin{cases} 2 & \text{if } |G| \text{ is odd;} \\ 2^k & \text{if } G \cong \mathbb{Z}_2^k \text{ and } G \cong \mathbb{Z}_2^{k-1} \times \mathbb{Z}_4; \\ 2^k + 1 & \text{if } G \cong \mathbb{Z}_4^k; \\ 2^{k+1} & \text{otherwise;} \end{cases}$$

Proof. **Case 1.** If $|G|$ is odd, then $G = Od(G)$and so $\deg(0) = |G| - 1$ and $\deg(x) = |G| - 2$ for all $x \in G \setminus \{0\}$. Hence $S = \{x, -x\}$ is a maximal independent set and so $\beta(\Gamma_{sq}(G)) = 2$.

Case 2. If $G \cong \mathbb{Z}_2^k$, then $Sq(G) = \{0\}^k$ and so $\Gamma_{sq}(G) \cong \bigcup_{|G|} K_1$. Hence $S = G$ is a maximal independent set and so $\beta(\Gamma_{sq}(G)) = 2^k$.

If $G \cong \mathbb{Z}_2^{k-1} \times \mathbb{Z}_4$,then $|G| = 2^{k+1}$. Here we have $|X_\ell| = 2$ for $1 \leq \ell \leq 2^k$. Hence $S = \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \times \{1, 3\}$ is a maximal independent set and so $\beta(\Gamma_{sq}(G)) = 2^k$.

Case 3. If $G \cong \mathbb{Z}_4^k$, then $|G| = 2^{2k}$. Let $X_1 = X_{(0,0,\ldots,0)} = Sq(G)$. By Remark 2.2, for $1 \leq i \neq j \leq 2^k$, $X_i \cap X_j = \emptyset$ and $|X_\ell| = \frac{|G|}{2^k}$, for $\ell, 2 \leq \ell \leq 2^k$. Then $\langle X_1 \rangle = K_{\frac{|G|}{2^k}}$ in $\Gamma_{sq}(G)$ and so for $x \in X_\ell$, then $\deg(x) = \frac{|G|}{2^k} - 2$ for $\ell, 2 \leq \ell \leq 2^k$. Hence $S = \{0\} \bigcup_{\ell=2}^{2^k} \{x, -x\}$ is a maximal independent set and so $\beta(\Gamma_{sq}(G)) = 2^k + 1$.

Case 4. For the remaining cases, $|X_\ell| \geq 3$ for $\ell, 2 \leq \ell \leq 2^k$. For each for $\ell, 2 \leq \ell \leq 2^k$, X_ℓ contains an element x such that $x \neq -x$. If $S = \{x_\ell, -x_\ell \in X_\ell\}$ for $1 \leq \ell \leq 2^k$. Hence S is a maximal independent set and so $\beta(\Gamma_{sq}(G)) = 2^{k+1}$. □

In the following theorem, we obtain the clique number of the $\Gamma_{sq}(G)$

Theorem 3.5. *Let G be a finite abelian group, $|G| = 2^{\alpha} \times p_1^{\alpha_1} \times \ldots \times p_r^{\alpha_r}$ where 2 and $p_i's$ are distinct primes and $\alpha, \alpha_i \in \mathbb{Z}^+ \cup \{0\}$ for $1 \leq i \leq r$ and $G \cong \prod_{i=1}^{k} \mathbb{Z}_{2^{m_i}} \times \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_r}$, $\alpha \in \mathbb{Z}^+$ and $\sum_{i=1}^{k} m_i = \alpha$. Then the clique number*

$$\omega(\Gamma_{sq}(G)) = \begin{cases} \frac{|G|+1}{2} & \textit{if } |G| \textit{ is odd;} \\ 1 & \textit{if } G \cong \mathbb{Z}_2^k; \\ 2 & \textit{if } G \cong \mathbb{Z}_2^{k-1} \times \mathbb{Z}_4; \\ \frac{|G|}{2^k} & \textit{if } G \cong \mathbb{Z}_4^k; \\ |T| + \frac{\frac{|G|}{2^k} - |T|}{2} & \textit{otherwise .} \end{cases}$$

Proof. **Case 1.** If $|G|$ is odd, then $G = Od(G)$and so $\deg(0) = |G| - 1$ and $\deg(x) = |G| - 2$ for all $x \in G \setminus \{0\}$. Let $S = \{x \text{ or} - x \ : x \in G\}$. Thus $\langle S \rangle$ is maximal complete subgraph of $\Gamma_{sq}(G)$. Hence $\omega(\Gamma_{sq}(G)) = |S| = \frac{|G|+1}{2}$.

Case 2. If $G \cong \mathbb{Z}_2^k$, then $Sq(G) = \{0\}^k$ and so $\Gamma_{sq}(G) \cong \bigcup\limits_{|G|} K_1$. Hence $S = \{x\}$ for all $x \in G$ is maximal complete subgraph of $\Gamma_{sq}(G)$ and so $\omega(\Gamma_{sq}(G)) = 1$.

Case 3. If $G \cong \mathbb{Z}_2^{k-1} \times \mathbb{Z}_4$,then $|G| = 2^{k+1}$. By Remark 2.2, here we have $|X_\ell| = 2$ and for $1 \leq i \neq j \leq 2^k$, $X_i \cap X_j = \emptyset$. Then $\langle X_\ell \rangle = K_2$ (or) $2K_1$ for $1 \leq \ell \leq 2^k$. Hence $S = \{x, y \in X_\ell : \ x + y \neq 0\}$ is maximal complete subgraph of $\Gamma_{sq}(G)$ and so $\omega(\Gamma_{sq}(G)) = 2$.

Case 4. If $G \cong \mathbb{Z}_4^k$, then $|G| = 2^{2k}$. Let $X_1 = X_{(0,0,\ldots,0)} = Sq(G)$. By Remark 2.2, for $1 \leq i \neq j \leq 2^k$, $X_i \cap X_j = \emptyset$ and $|X_\ell| = \frac{|G|}{2^k}$, for $\ell, 2 \leq \ell \leq 2^k$. Then $\langle X_1 \rangle = K_{\frac{|G|}{2^k}}$ in $\Gamma_{sq}(G)$ and so for $x \in X_\ell$, then $\deg(x) = \frac{|G|}{2^k} - 2$ for $\ell, 2 \leq \ell \leq 2^k$. Hence $S = X_1$ is a maximal complete subgraph of $\Gamma_{sq}(G)$ and so $\omega(\Gamma_{sq}(G)) = \frac{|G|}{2^k}$.

Case 5. Since $\Gamma_{sq}(G)$ is a disjoint union of 2^k connected induced subraph of $\langle X_\ell \rangle$. From this, we get that $\omega(\Gamma_{sq}(G)) = \ \max\{\omega(\langle X_1 \rangle), \omega(\langle X_2 \rangle), \ldots, \omega(\langle X_{2^k} \rangle)\}$.

In the remaining cases, G is of the form $G = \mathbb{Z}_2^q \times \mathbb{Z}_{2^{\beta_{q+1}}} \times \ldots \times \mathbb{Z}_{2^{\beta_k}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \ldots \times \mathbb{Z}_{p_r^{\alpha_r}}$, where $k \geq 0$, $q \geq 0$, and at least one $\alpha_i \geq 1$ for $1 \leq i \leq r$ or at least one $\beta_i \geq 2$ for $0 \leq i \leq k$.

Let S be a maximal complete subgraph of $\Gamma_{sq}(G)$. Since $\deg(x) \geq \deg(y)$ for all $x \in X_1$, $y \in X_\ell$, $|X_\ell| = \frac{|G|}{2^k} \geq 3$ and $\Gamma_{sq}(G)$ is a disjoint union of $\langle X_\ell \rangle$, for $1 \leq \ell \leq 2^k$. Then S must in X_1.

Let $S \subseteq X_1 = X_{(0,0,\ldots,0)}$. If $T = \{x \in X_1 \ |2x = 0\} \cup \{0\}$, then $|T| = 2^{k-q}$ and $\langle T \rangle = K_{|T|}$ in $\Gamma_{sq}(G)$ and let $T^c = X_1 \setminus T$. If for any $x \in T^c$, then x is not adjacent to $-x$ and so $\deg(x) = |T^c| - 2$. Thus $S = T \cup \{x \ \text{ or } \ -x \ |x \in T^c\}$. This S is a maximal complete subgraph of $\langle X_1 \rangle$ and $|S| = |T| + \frac{\frac{|G|}{2^k} - |T|}{2}$. Hence S is a maximal complete subgraph of $\Gamma_{sq}(G)$ and so $\omega(\Gamma_{sq}(G)) = |T| + \frac{\frac{|G|}{2^k} - |T|}{2}$.

□

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