International Journal of Computer Science
Scholarly Peer Reviewed Research Journal - PRESS - OPEN ACCESS
ISSN: 2348-6600
Since 2012

http://www.ijcsjournal.com Volume 8, Issue 2, No 03, 2020 ISSN: 2348-6600
Reference ID: IJCS-368 PAGE NO: 2489-2499

Department of Computer Science, DDE, Madurai Kamaraj University, India 23rd to 25th September 2020
1st International E-Conference on Recent Developments in Science, Engineering and Information Technology (ICRDSEIT–2020)
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Numerical solution of boundary value problems using Hermite wavelet-Galerkin method.

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Abstract

In this paper, we proposed Numerical solution of boundary value problems using Hermite Wavelet-Galerkin Method. Here, we used weight functions as a Hermite wavelets that are assumed basis elements which allow us to obtain the numerical solutions of boundary value problems. The numerical solutions obtained by this method are compared with the exact solution and existing methods. Some of the problems are given to demonstrate the effectiveness and accuracy of the proposed method.

Keywords: Hermite wavelets; Galerkin method; Numerical solution; Finite difference method.

1. Introduction

It has been observed from the literature that many researchers are developing fast and accurate numerical methods to handle the different problems arising in various fields of science and engineering. In the past finite difference methods (FDM) and finite element methods (FEM) were commonly used for solving such type of problems [1]. A boundary value problem (BVP) is a differenntial equation together with a set of boundary conditions. A solution to a BVP is a solution to the differential equation which also satisfies the boundary conditions. In general it is not possible to obtain analaytical solution of an arbitrary boundary value problem. This necessitates either discretization of boundary value problems leading to numerical solutions. The Galerkin method is one of the best known methods for finding numerical solutions of such type of problems and is considered the most widely used in applied mathematics [2]. Its simplicity makes it perfect for many applications. The wavelet-Galerkin method is an improvement over the standard Galerkin methods. The advantage of wavelet-Galerkin method over finite difference or finite element method has lead to tremendous applications in science and engineering. An approach to study

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differential equations is the use of wavelet function bases in place of other conventional piecewise polynomial trial functions in finite element type methods.

Wavelets theory has been received a much interest because of the comprehensive mathematical power and the good application potential of wavelets in science and engineering problems. Wavelets have generated significant interest from both theoretical and applied researchers over the last few decades. The concepts for understanding wavelets were provided by Meyer, Mallat, Daubechies, and many others, [3]. Since then, the number of applications where wavelets have been used has exploded. In areas such as approximation theory and numerical solutions of differential equations, wavelets are recognized as powerful weapons not just tools.

In this paper, we developed numerical solution of boundary value problems using Hermite wavelet-Galerkin method (HWGM). This method is based on expanding the solution by Hermite wavelets with unknown coefficients. The properties of Hermite wavelets together with the Galerkin method are utilized to evaluate the unknown coefficients and then a numerical solution of boundary value problems is obtained.

The present paper is organized as follows; Preliminaries of Hermite wavelets are presented in section 2. In section 3, method of solution is discussed. Section 4 deals with the numerical findings and error analysis. Finally, conclusion of the proposed work is discussed in section 5.

2. Preliminaries of Hermite wavelets

Wavelets form a family of functions which are generated from dilation and translation of a single function which is called as mother wavelet $\psi(x)$. If the dialation parameter a and translation parameter b varies continuously, we have the following family of continuous wavelets [4 , 5]:

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$$\psi_{a,b}(x) = |a|^{-1/2}\,\psi\left(\frac{x-b}{a}\right),\ \forall\ a,b\in R,\ \ a\neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0\,a_0^{-k}$, $a_0 > 1$, $b_0 > 0$. We have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{1/2}\,\psi(a_0^k\,x - nb_0),\ \forall\ a,\ b\in R,\ \ a\neq 0,$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$,then $\psi_{k,n}(x)$ forms an orthonormal basis. Hermite wavelets are defined as

$$\psi_{n,m}(x) = \begin{cases} \dfrac{2^{\frac{k}{2}}}{\sqrt{\pi}}\tilde{H}_m(2^k x - 2n + 1), & \dfrac{n-1}{2^{k-1}} \leq x < \dfrac{n}{2^{k-1}} \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where $$\tilde{H}_m = \sqrt{\frac{2}{\pi}}\,H_m(x) \tag{2.2}$$

where $m = 0,1,\ldots,M-1$. In eq. (2.2) the coefficients are used for orthonormality. Here $H_m(x)$ are the second Hermite polynomials of degree m with respect to weight function $W(x) = \sqrt{1-x^2}$ on the real line R and satisfies the following reccurence formula $H_0(x) = 1$, $H_1(x) = 2x$,

$$H_{m+2}(x) = 2xH_{m+1}(x) - 2(m+1)H_m(x),\ \text{ where } m = 0,1,2,\ldots. \tag{2.3}$$

For $k = 1$ & $n = 1$in (2.1) and (2.2), then the Hermite wavelets are given by

$$\psi_{1,0}(x) = \frac{2}{\sqrt{\pi}},\quad \psi_{1,1}(x) = \frac{2}{\sqrt{\pi}}(4x-2),\quad \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(16x^2 - 16x + 2),$$

$$\psi_{1,3}(x) = \frac{2}{\sqrt{\pi}}(64x^3 - 96x^2 + 36x - 2),\quad \psi_{1,4}(x) = \frac{2}{\sqrt{\pi}}(256x^4 - 512x^3 + 320x^2 - 64x + 2)\ \text{, and so on.}$$

Function approximation:

We would like to bring a solution function $u(x)$ under Hermite space by approximating $u(x)$ by elements of Hermite wavelet bases as follows,

$$u(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\,\psi_{n,m}(x) \tag{2.4}$$

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where $\psi_{n,m}(x)$ is given in eq. (2.1).

We approximate $u(x)$ by truncating the series represented in Eq. (2.4) as,

$$u(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\, \psi_{n,m}(x) \tag{2.5}$$

where c and ψ are $2^{k-1} M \times 1$ matrix.

Convergence of Hermite wavelets

Theorem: If a continuous function $u(x) \in L^2(R)$ defined on $[0,1)$ be bounded, i.e. $u(x) \le K$, then the Hermite wavelets expansion of $u(x)$ converges uniformly to it [6].

3. Method of Solution

Consider the one dimensional partial differential equation is of the form,

$$\frac{d^2 y}{d x^2} + \alpha \frac{d y}{d x} + \beta u = f(x) \tag{3.1}$$

With boundary conditions $y(0) = a, \quad y(1) = b$ (3.2)

Where α, β are may be a functions of x or constant and $f(x)$ be a continuous function.

Write the equation (4.1) as

$$R(x) = \frac{d^2 y}{d x^2} + \alpha \frac{d y}{d x} + \beta u - f(x) \tag{3.3}$$

where $R(x)$ is the residual of the eq. (4.1). When $R(x) = 0$ for the exact solution, $u(x)$ only which will satisfy the boundary conditions.

Consider the trail series solution of the differential equation (3.1), $y(x)$ defined over [0, 1) can be expanded as a modified Laguerre wavelet, satisfying the given boundary conditions which is involving unknown parameter as follows,

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$$y(x) = \sum_{i=1}^{2^{k-1}} \sum_{j=1}^{M} c_{i,j}\, \psi_{i,j}(x) \tag{3.4}$$

where $c_{i,j}$'s are unknown coefficients to be determined.

Accuracy in the solution is increased by choosing higher degree Laguerre wavelet polynomials.

Differentiating eq. (3.4) twice with respect to x and substitute the values of in $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, y eq. (3.3). To find $c_{i,j}$'s we choose weight functions as assumed bases elements and integrate on boundary values together with the residual to zero [8].

i.e. $$\int_0^1 \psi_{1,j}(x)\; R(x)\; dx = 0,\; j = 1, 2, \ldots\ldots n$$

then we obtain a system of linear equations, on solving this system, we get unknown parameters. Then substitute these unknowns in the trail solution, numerical solution of eq. (3.1) is obtained.

In order to know the accuracy of HWGM for the test problems, we use the error measure i.e. maximum absolute error. The maximum absolute error will be calculated by

$$E_{\max} = \max \left| \, y(x,\, t)_e - u(x,\, t)_a \, \right|,$$

4. Numerical Implementation:

In this section, we apply HWGM discussed in section 3 to some of the boundary value problems.

Problem 4.1 First, consider the BVP [8],

$$\frac{d^2y}{dx^2} + y = -x,\; 0 \le x \le 1 \tag{4.1}$$

With boundary conditions: $y(0) = 0,\; y(1) = 0$ (4.2)

The implementation of the eq. (4.1) as per the method explained in section 3 is as follows:

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The residual of eq. (4.1) can be written as: $R(x) = \frac{d^2 y}{d x^2} + y + x$ (4.3)

Now choosing the weight function $w(x) = x(1-x)$ for Hermite wavelet bases to satisfy the given boundary conditions (4.2), i.e. $\psi(x) = w(x) \times \psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1-x) = \frac{2}{\sqrt{\pi}} x(1-x) \quad , \quad \psi_{1,1}(x) = \psi_{1,1}(x) \times x(1-x) = \frac{2}{\sqrt{\pi}}(4x-2)x(1-x)$$

$$\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1-x) = \frac{2}{\sqrt{\pi}}(16x^2 - 16x + 2)x(1-x)$$

Assuming the trail solution of (5.1) for $k = 1$ and $m = 3$ is given by

$$y(x) = c_{1,0}\,\psi_{1,0}(x) + c_{1,1}\,\psi_{1,1}(x) + c_{1,2}\,\psi_{1,2}(x) \tag{4.4}$$

Then the eq. (4.4) becomes

$$y(x) = c_{1,0}\frac{2}{\sqrt{\pi}}x(1-x) + c_{1,1}\frac{2}{\sqrt{\pi}}(4x-2)x(1-x) + c_{1,2}\frac{2}{\sqrt{\pi}}(16x^2 - 16x + 2)x(1-x) \tag{4.5}$$

Differentiating eq. (4.5) twice w.r.t. x we get,

i.e. $$\frac{dy}{dx} = c_{1,0}\frac{2}{\sqrt{\pi}}(1-2x) + c_{1,1}\frac{2}{\sqrt{\pi}}(-12x^2 + 12x - 2) + c_{1,2}\frac{2}{\sqrt{\pi}}(-64x^3 + 96x^2 - 36x + 2) \tag{4.6}$$

$$\frac{d^2 y}{d x^2} = c_{1,0}\frac{2}{\sqrt{\pi}}(-2) + c_{1,1}\frac{2}{\sqrt{\pi}}(-24x + 12) + c_{1,2}\frac{2}{\sqrt{\pi}}(-192x^2 + 192x - 36) \tag{4.7}$$

Using eq. (4.5) and (4.7), then eq. (4.3) becomes,

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$$R(x) = c_{1,0}\frac{2}{\sqrt{\pi}}(-2) + c_{1,1}\frac{2}{\sqrt{\pi}}(-24x+12) + c_{1,2}\frac{2}{\sqrt{\pi}}(-192x^2+192x-36) + \left(c_{1,0}\frac{2}{\sqrt{\pi}}x(1-x) + c_{1,1}\frac{2}{\sqrt{\pi}}(4x-2)x(1-x) + c_{1,2}\frac{2}{\sqrt{\pi}}(16x^2-16x+2)\right) + x$$

$$\Rightarrow R(x) = c_{1,0}\frac{2}{\sqrt{\pi}}(-x^2+x-2) + c_{1,1}\frac{2}{\sqrt{\pi}}(-4x^3+6x^2-26x+12) + c_{1,2}\frac{2}{\sqrt{\pi}}(-16x^4+32x^3-210x^2+194x-36) + x \quad (4.8)$$

This is the residual of eq. (4.1). The "weight functions" are the same as the bases functions. Then by the weighted Galerkin method, we consider the following:

$$\int_0^1 \psi_{1,m}(x)R(x)dx = 0, \; m = 0,1,2 \quad (4.9)$$

For $m = 0, 1, 2$ in eq. (4.9),

$$\text{i.e. } \int_0^1 \psi_{1,0}(x)R(x)dx = 0, \; \int_0^1 \psi_{1,1}(x)R(x)dx = 0, \; \int_0^1 \psi_{1,2}(x)R(x)dx = 0$$

$$\Rightarrow \quad (-0.3802)c_{1,1} + (0)c_{1,2} + (0.4487)c_{1,3} + 0.0940 = 0 \quad (4.10)$$

$$(0)c_{1,1} - (0.9943)c_{1,2} + (0)c_{1,3} + 0.0376 = 0 \quad (4.11)$$

$$(0.4487)c_{1,0} + (0)c_{1,1} - (2.3686)c_{1,2} - 0.1128 = 0 \quad (4.12)$$

We have three equations (4.10) – (4.12) with three unknown coefficients i.e. $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. By solving this system of algebraic equations, we obtain the values of $c_{1,0} = 0.2446$, $c_{1,1} = 0.0378$ and $c_{1,2} = -0.0013$. Substituting these values in eq. (4.5), we get the numerical solution.

Obtained numerical solutions are compared with exact solution $y(x) = \frac{\sin(x)}{\sin(1)} - x$ and other existing method solutions are presented in figure 1 and table 1.

Table 1. Comparison of numerical solution and exact solution of the problem 4.1

x	Numerical solution		Exact solution	Absolute error	
	FDM	HWGM		FDM	HWGM
0.1	0.018660	0.018624	0.018642	1.80e-05	1.80e-05
0.2	0.036132	0.036102	0.036098	3.40e-05	4.00e-06
0.3	0.051243	0.051214	0.051195	4.80e-05	1.90e-05

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0.4	0.062842	0.062793	0.062783	5.90e-05	1.00e-05
0.5	0.069812	0.069734	0.069747	6.50e-05	1.30e-05
0.6	0.071084	0.070983	0.071018	6.60e-05	3.50e-05
0.7	0.065646	0.065545	0.065585	6.10e-05	4.00e-05
0.8	0.052550	0.052481	0.052502	4.80e-05	2.10e-05
0.9	0.030930	0.030908	0.030902	2.80e-05	6.00e-06

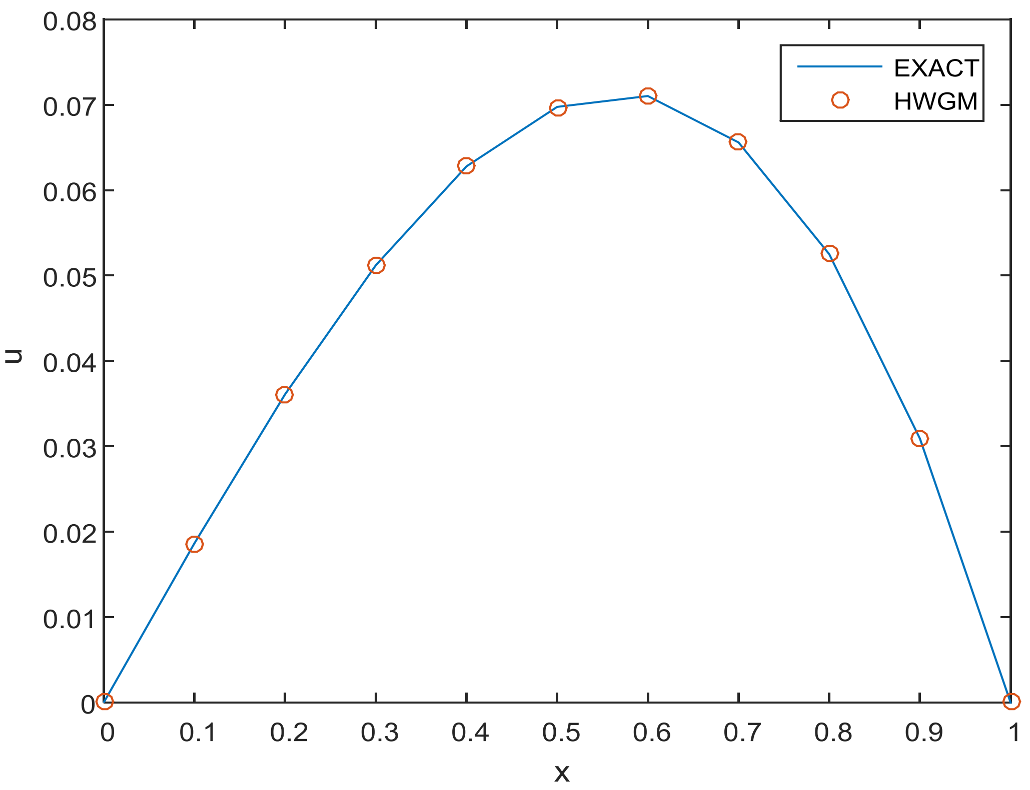

Fig. 1. Comparison of numerical and exact solutions of the problem 4.1.

Problem 4.2 Next, consider singular boundary value problem [9]

$$\frac{d^2 y}{d x^2} + \frac{8}{x}\frac{d y}{d x} + x y = x^5 - x^4 + 44x^2 - 30x, \quad 0 \le x \le 1 \qquad (4.13)$$

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With boundary conditions: $y(0) = 0, \quad y(1) = 0$ (4.14)

Which has the exact solution $y(x) = -x^3 + x^4$.

By applying the method explained in the section 3, we obtain the constants and substituting these values in eq. (4.5) we get the numerical solution. Obtained numerical solutions are compared with exact and other existing method solutions are presented in table 2 and figure 2.

Table 2. Comparison of numerical solution and exact solution of the problem 4.2.

x	Numerical solution		Exact solution	Absolute error	
	FDM	HWGM		FDM	HWGM
0.1	0.024647	-0.000900	-0.000900	2.55e-02	0
0.2	0.024538	-0.006401	-0.006400	3.09e-02	1.00e-06
0.3	0.016024	-0.018904	-0.018900	3.40e-02	4.00e-06
0.4	-0.000072	-0.038407	-0.038400	3.83e-02	7.00e-06
0.5	-0.022021	-0.062512	-0.062500	4.05e-02	1.20e-05
0.6	-0.045926	-0.086417	-0.086400	4.05e-02	1.70e-05
0.7	-0.065532	-0.102920	-0.102900	3.74e-02	2.00e-05
0.8	-0.072190	-0.102420	-0.102400	3.02e-02	2.00e-05
0.9	-0.054840	-0.072914	-0.072900	1.81e-02	1.40e-05

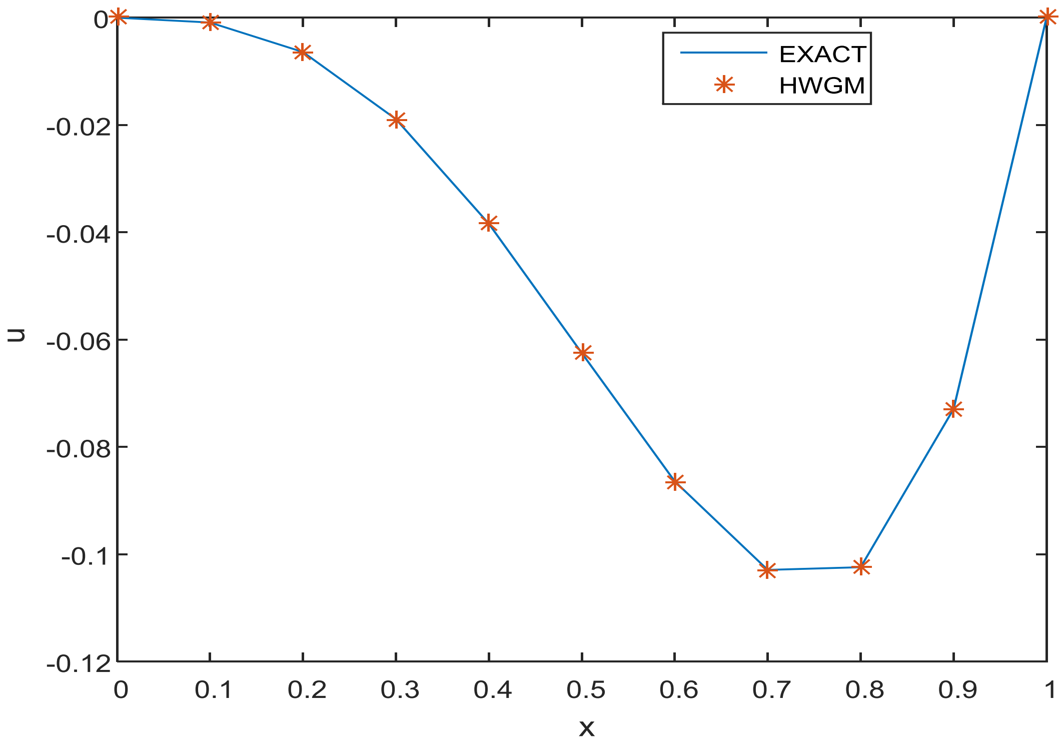

Fig. 2. Comparison of numerical and exact solutions of the problem 4.2.

5. Conclusion

In this paper, we proposed the Numerical Solution of Boundary Value Problems Using Hermite Wavelet-Galerkin Method.. The efficiency of the method is observed through the test problems and the numerical solutions are presented in Tables and figures, which show that HWGM gives comparable results with the exact solution and better than existing numerical methods. Hence the proposed method is effective for solving boundary value problems.

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