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Results on annihilator graph of a commutative Ring

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Abstract

Let R be a commutative ring with identity 1. $Z(R)$ be its set of zero-divisors, and if $a \in Z(R)$, then let $ann(a) = \{d \in R | da = 0\}$. The annihilator graph R is the (undirected) graph $AGC(R)$ with vertices $Z(R)^* = Z(R) - \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann(x) \neq ann(y)$. In this article, we study the graph $AGC(R)$. For a commutative ring R, we show that $AGC(R)$ is connected with diameter at most two and with girth at most four provided that $AGC(R)$ has a cycle.

Key Words: Annihilator Graph; diameter; girth; Zero divisor Graph.
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0.1 INTRODUCTION

Let R be a commutative ring with identity 1, and let $Z(R)$ be its set of zero divisors. Probably the most attention has been to the *zero divisor graph* $\Gamma(R)$ for a commutative ring R. The set of vertices of $\Gamma(R)$ is $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The zero-divisor graph was introduced by David F. Anderson and Paul S. Livingston in [1]. In this article, we introduce the *annihilator graph* $AGC(R)$ for a commutative ring R. Let $a \in Z(R)$ and let $ann(a) = \{d \in R | da = 0\}$. The annihilator graph of R is the (undirected) graph $AGC(R)$ with vertices $Z(R)^* = Z(R) - \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann(x) \neq ann(y)$.

In the second section, we show that $AGC(R)$ is connected with diameter at most two. Also, we determine when $AGC(R)$ is a complete graph, or a star graph.

Let G be a (undirected) graph. We say that G is *connected* if there is a path between any two distinct vertices. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y (($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path). Then the *diameter* of G is $diam(G)$ =sup$\{d(x, y)|x$ and y are vertices of $G\}$. The *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in $G(gr(G) = \infty)$ if G contains no cycles.

A graph G is *complete* if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K^n (we allow n to be an infinite cardinal). A *complete bipartite* graph is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets.

Throughout, R will be a commutative ring with nonzero identity, $Z(R)$ its set of zero divisors, $Nil(R)$ its set of nilpotent elements. $U(R)$ its group of units. $T(R)$ its total quotient ring, and $Min(R)$ its set of minimal prime ideals. We say that R is *reduced* if $Nil(R) = \{0\}$.

0.2 BASIC PROPERTIES OF AGC(R)

In this section, we show that $AGC(R)$ is connected with diameter at most two. If $AGC(R) \neq \Gamma(R)$, we show that $gr(AGC(R)) \in \{3, 4\}$.

Theorem 0.2.1. *[3, Theorem 3.13] Let R be a nonreduced commutative ring with $|Nil(R)^*| \geq 2$, and let $\Gamma_N(R)$ be the induced subgraph of $\Gamma(R)$ with vertices $Nil(R)^*$. Then $\Gamma_N(R)$ is complete if and only if $Nil(R)^2 = 0$.*

Lemma 0.2.2. *Let R be a nonreduced commutative ring with $|Z(R)^*| = |Nil(R)^*| \geq 2$. Then $AGC(R)$ is disconnected if and only if $Nil(R)^2 = \{0\}$.*

Proof. ($\Rightarrow$) Assume that $AGC(R)$ is disconnected. Suppose $Nil(R)^2 \neq 0$. Let $x, y, z \in Z(R)^*$ and assume that $x^2 \neq 0$. Since x is a zero divisor, then there

is a $y \in Z(R)^*$ such that $xy = 0$. Therefore $y \in ann(x)$ and $x \in ann(y)$, but $x \notin ann(x)$. Thus $ann(x) \neq ann(y)$ and hence $x - y$ is an edge of $AGC(R)$, which is a condradiction. Thus $Nil(R)^2 = \{0\}$.
$(\Leftarrow)$ If $Nil(R)^2 = \{0\}$.
Case 1. Suppose $|Z(R)^*| = |Nil(R)^*| = 2$. Let $a, b \in Z(R)^*$ such that $ab = 0$. Since $Nil(R)^2 = 0$. Then $ann(a) = ann(b)$. Hence $a - b$ is not an edge in $AGC(R)$.
Case 2. Suppose $|Z(R)^*| = |Nil(R)^*| \geq 3$. Let $a, b, c \in Z(R)^*$. Since $Nil(R)^2 = 0$, then $\Gamma_N(R)$ is complete by Theorem 0.2.1. So that $a - b - c - a$ are adjacent in $\Gamma(R)$ [Since$|Z(R)^*| = |Nil(R)^*|$]. Thus $ann(a) = ann(b) = ann(c)$ and hence $a - b, b - c, c - a$ are not adjacent in $AGC(R)$.
In both cases $AGC(R)$ is disconnected. □

The following is an example of disconnected graph.

Example 0.2.3. *Let* $R = \frac{\mathbb{Z}_2[x,y]}{<x,y>^2}$. *Then* $\Gamma(R) = K_3$ *and* $AGC(R) = \overline{K}_3$.

The following results are true except in the case of Theorem 0.2.2.

Lemma 0.2.4. *If* $x - y$ *is an edge of* $\Gamma(R)$*, then* $x - y$ *is an edge of* $AGC(R)$*. In particular* P *is a path in* $\Gamma(R)$*, then* P *is also a path in* $AGC(R)$*.*

Proof. Given, $x - y$ is an edge of $AGC(R)$, then $xy = 0$. Therefore $y \in ann(x)$ and $x \in ann(y)$. But $x \notin ann(x)$ and $y \notin ann(y)$. Therefore $ann(x) \neq ann(y)$. Hence $x - y$ is an edge of $AGC(R)$. Suppose $x - y - z$ is a path in $\Gamma(R)$, then $x - y - z$ is also a path in $AGC(R)$. □

Lemma 0.2.5. *(1) If* $ann(x) \subset ann(y)$ *or* $ann(y) \subset ann(x)$ *for some distinct* $x, y \in Z(R)^*$*, then* $x - y$ *is an edge of* $AGC(R)$*.*
(2) If $ann(x) \nsubseteq ann(y)$ *or* $ann(y) \nsubseteq ann(x)$ *for some distinct* $x, y \in Z(R)^*$*, then* $x - y$ *is an edge of* $AGC(R)$*.*
(3) If $d_{\Gamma(R)}(x, y) = 3$ *for some distinct* $x, y \in Z(R)^*$*, then* $x - y$ *is an edge of* $AGC(R)$*.*

Proof. (1) Since $ann(x) \subset ann(y)$, then there exixts an element $a \in ann(y)$, and $a \notin ann(x)$ such that $ann(x) \neq ann(y)$. Hence $x - y$ is an edge of $AGC(R)$.

(2) Suppose $ann(x) \nsubseteq ann(y)$ or $ann(y) \nsubseteq ann(x)$, then $ann(x) \neq ann(y)$. Hence $x - y$ is an edge of $AGC(R)$.

(3) Let x and y be distinct vertices in $Z(R)^*$. Given $d_{\Gamma(R)}(x, y) = 3$. Let us assume that $x - a - b - y$ is a shortest path connecting x and y in $\Gamma(R)$ where a, b are distinct vertices in $Z(R)^*$. We have $xa = 0, ab = 0, by = 0$. Therefore $a \in ann(x)$ and $a \notin ann(y)$. This implies that $ann(x) \neq ann(y)$ and hence $x - y$ is an edge of $AGC(R)$. □

Theorem 0.2.6. *Let R be a commutative ring. Suppose that $d_{\Gamma(R)}(x, y) = 3$ for some distinct $x, y \in Z(R)^*$. Then there exists a cycle of length 3 in $AGC(R)$ and at least one edge of C is an edge of $\Gamma(R)$.*

Proof. Given $d_{\Gamma(R)}(x, y) = 3$ for some distinct $x, y \in Z(R)^*$. Then there exists a path from $x - a - b - y$ in $\Gamma(R)$, where $a, b \in Z(R)^*$ and $a \neq b$. In this, $ann(x) \nsubseteq ann(y)$ and $ann(y) \nsubseteq ann(x)$. Then $x - y$ is an edge of $AGC(R)$ and $bx \neq 0$. So $b \notin ann(x)$ and $x \notin ann(b)$. Therefore $ann(x) \neq ann(b)$. Then $x - b$ is an edge of $AGC(R)$. Hence $C : x - b - y - x$ is a cycle of length 3 in $AGC(R)$ and at least one edge of C is an edge of $\Gamma(R)$. □

Theorem 0.2.7. *Let R be a reduced commutative ring. Suppose that $x - y$ is an edge of $AGC(R)$ that is not an edge of $\Gamma(R)$. Then there is a cycle of length 3 in $AGC(R)$ and at least one edge of C is an edge of $\Gamma(R)$.*

Proof. Suppose that $x - y$ is an edge of $AGC(R)$ that is not an edge of $\Gamma(R)$. Then $ann(x) \neq ann(y)$ such that $a \in ann(x)$ and $a \notin ann(y)$. Then $ax = 0$. Since R is reduced $a \neq x$ and y is a zero divisor. Then there is $b \neq a \in Z(R)^*$ such that $by = 0$. Since R is reduced, so that $b \neq y$. Hence $x - b$ is an edge of $AGC(R)$, we have $x - y - b - x$ is a cycle of length 3 in $AGC(R)$ and at least one edge of C is an edge of $\Gamma(R)$. □

Theorem 0.2.8. *If $x - y$ is not an edge of $AGC(R)$ for some distinct $x, y \in Z(R)^*$, then there is a $w \in Z(R)^* - \{x, y\}$ such that $x - w - y$ is a path in $\Gamma(R)$ and hence $x - w - y$ is also a path in $AGC(R)$.*

Proof. Given, $x - y$ is not an edge of $AGC(R)$ for some distinct $x, y \in Z(R)^*$. Therefore $ann(x) = ann(y)$. Then there exists an element $w \in ann(x) = ann(y)$ such that $xw = yw = 0$. We conclude that $x - w - y$ is a path in $\Gamma(R)$. Hence by Lemma 0.2.4 $x - w - y$ is also a path in $AGC(R)$. □

Theorem 0.2.9. *Let R be a commutative ring with $|Z(R)|^* \geq 2$. Then $AGC(R)$ is connected and $diam(AGC(R)) \leq 2$.*

Proof. **Case 1.**If $|Z(R)^*| = 2$. Let $x, y \in Z(R)^*$. Then $xy = 0$. Hence $x - y$ is an edge of $\Gamma(R)$. By Lemma 0.2.4 $x - y$ is an edge of $AGC(R)$.
Case 2. If $|Z(R)^*| > 2$. Let $x, y, z \in Z(R)^*$. Suppose $xy \neq 0$, using Lemma 0.2.4, we get the result. □

0.3 When $AGC(R) = \Gamma(R)$?

Theorem 0.3.1. *[1,Theorem 2.5] Let R be a commutative ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R = \mathbb{Z}_2 \times A$ where A is an integral domain, or $Z(R)$ is an annihilator ideal (and hence is prime).*

Theorem 0.3.2. *Let R be a commutative ring with identity 1. Then there is a vertex $x \in Z(R)^*$ such that x is adjacent to all vertices in $AGC(R)$ and hence $AGC(R) = \Gamma(R)$ if and only if $R = \mathbb{Z}_2 \times A$ where A is an integral domain.*

Proof. ($\Rightarrow$). Since $AGC(R) = \Gamma(R)$, using Theorem 0.3.1 we get $R = \mathbb{Z}_2 \times A$.
($\Leftarrow$). Given, $R = \mathbb{Z}_2 \times A$ where A is an integral domain. Using Theorem 0.3.1, there is a vertex which is adjacent to every other vertex in $\Gamma(R)$. Let $x \in Z(R)^*$ such that x is adjacent to all vertices of $\Gamma(R)$. Clearly $ann(x) = Z(R) - \{x\}$ for some $x \in Z(R)^*$, then x is adjacent to every other vertex. Thus $ann(y) = \{0, x\}$ for every $y - \{x\} \in Z(R)^*$. Therefore $ann(y) = ann(z)$ for every $z \in Z(R)^* - \{x\}$. Hence no two elements in $Z(R)^* - \{x\}$ are not adjacent in $AGC(R)$ and hence $AGC(R) = \Gamma(R)$.

□

Theorem 0.3.3. *[4, Theorem 2.8] Let R be a commutative ring. Then $diam(\Gamma(R)) = 2$ if and only if either (a) R is reduced with exactly two minimal primes and at least three nonzero divisors, or (b) $Z(R)$ is an ideal whose square is not $\{0\}$ and each pair of distinct zero divisors has a nonzero annihilator.*

Theorem 0.3.4. *[3, Theorem 3.2] Let R be a reduced commutative ring that is not an integral domain, and let $z \in Z(R)^*$. Then:*

(a) $ann_R(z) = ann_R(z^n)$ for each positive integer $n \geq 2$;

(b) If $c + z \in Z(R)$ for some $c \in ann_R(z) \setminus \{0\}$, then $ann_R(z + c)$ is properly contained in $ann_R(z)$ (i.e., $ann_R(c + z) \subset ann_R(z)$). In particular, if $Z(R)$ is an ideal of R and $c \in ann_R(z) \setminus \{0\}$, then $ann_R(z + c)$ is properly contained in $ann_R(z)$.

Theorem 0.3.5. *Let R be a reduced commutative ring that is not an integral domain. Then the following statements are equivalent.*
(a) $AGC(R)$ is complete;
(b) $\Gamma(R)$ is complete;
(c) R is ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. $(a) \Rightarrow (b)$. Let $b \in Z(R)^*$. Suppose that $b^2 \neq a$. Since $ann(b) = ann(b^2)$, So that $b - b^2$ is not an edge of $AGC(R)$, a condradiction. Thus $b^2 = b$ for each $b \in Z(R)^*$. Let x, y be two distinct elements in $Z(R)^*$. To prove $x - y$ is an edge of $\Gamma(R)$. Suppose that $xy \neq 0$. Since $x - y$ is an edge of $AGC(R)$, we have $xy \neq 0$. Now, $ann(x(xy)) = ann(x^2y) = ann(xy)$. Thus $xy - x$ is not an edge of $AGC(R)$, a condradiction. Hence $xy = 0$ and $x - y$ is an edge of $\Gamma(R)$. $(b) \Rightarrow (c)$ It follows from Theorem 0.3.3.
$(c) \Rightarrow (a)$ Given, $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, which is complete. Hence $AGC(R)$ is complete. □

Theorem 0.3.6. *Let R be a reduced commutative ring that is not an integral domain and assume that $Z(R)$ is an ideal of R. Then $AGC(R) \neq \Gamma(R)$ and $gr(AGC(R)) = 3$.*

Proof. Let $z \in Z(R)^*$, $a \in ann(z) - \{0\}$, and $h \in ann(a+z) - \{0\}$. Then $h \in ann(a+z) \subset ann(z)$ by Theorem 0.3.4 and thus $ha = 0$, since $a^2 \neq 0$, we have $h \neq a$, and hence $h+z \neq a+z$, since $\{h, a\} \subseteq ann(z)$ and $z^2 \neq 0$, we have $(h+z)$ and $(a+z)$ are non zero distinct elements in $Z(R)^*$. Since $(h+z)(a+z) = z^2 \neq 0$, we have $(h+z)-(a+z)$ is not an edge of $\Gamma(R)$. Since $a^2 \neq 0$ and $h^2 \neq 0$, so $(h+z)-(a+z)$ is an edge of $AGC(R)$. Thus $AGC(R) \neq \Gamma(R)$ and $h - a - (h+z) - h$ is a cycle of length 3 in $AGC(R)$ and $gr(AGC(R)) = 3$. □

Theorem 0.3.7. *Let R be a reduced commutative ring with $|Min(R)| \geq 3$(possibly $Min(R)$is infinite).Then $AGC(R) \neq \Gamma(R)$ and $gr(AGC(R)) = 3$.*

Proof. If $Z(R)$ is an ideal of R, then $AGC(R) \neq \Gamma(R)$ by Theorem 0.3.6. Hence assume that $Z(R)$ is not an ideal of R. Since $|Min(R)| \geq 3$, we have $diam(\Gamma(R)) = 3$ by Theorem 0.3.3 and by Theorem 0.2.9 $AGC(R) \neq \Gamma(R)$. Since R is reduced and $AGC(R) \neq \Gamma(R)$, we have $gr(AGC(R)) = 3$ by Theorem 0.2.7. □

Theorem 0.3.8. *Let R be a reduced commutative ring that is not an integral domain. Then $AGC(R) = \Gamma(R)$ if and only if $|Min(R)| = 2$.*

Proof. Assume that $AGC(R) = \Gamma(R)$. Since R is reduced commutative ring that is not an integral domain, $|Min(R)| = 2$ by Theorem 0.3.7. Conversely, assume that $|Min(R)| = 2$. Let p_1, p_2 minimal prime ideal of R. Since R is reduced, we have $Z(R) = p_1 \cup p_2$ and $p_1 \cap p_2 = \{0\}$. Let $x, y \in Z(R)^*$. Assume that $x, y \in p_1$. Since $p_1 \cap p_2 = \{0\}$, neither $x \in p_2$ nor $y \in p_2$, and thus $xy \neq 0$. Since $p_1 p_2 \subseteq p_1 \cap p_2 = \{0\}$, it follows that $ann(x) = ann(y) = p_2$. Thus $x - y$ is not an edge of $AGC(R)$. Similarly if $x, y \in p_2$, then $x - y$ is not an edge of $AGC(R)$. Hence each edge of $AGC(R)$ is an edge of $\Gamma(R)$, and $AGC(R) = \Gamma(R)$. □

Theorem 0.3.9. *[2, Theorem 2.2] The following statement are equivalent for a reduced commutative ring R.*
(1) $gr(\Gamma(R)) = 4$.
(2) $T(R) = F_1 \times F_2$*, where each* F_i *is a field with* $|F_i| \geq 3$.
(3) $\Gamma(R) = K_{m,n}$ *with* $m, n \geq 2$.

Theorem 0.3.10. *Let R be a reduced commutative ring. Then the following statements are equivalent:*
(1) $gr(AGC(R)) = 4$;
(2) $AGC(R) = \Gamma(R)$ *and* $gr(\Gamma(R)) = 4$;
(3) $gr(\Gamma(R)) = 4$;
(4) $T(R)$ *is ring -isomorphic to* $F_1 \times F_2$*, where each* F_i *is a field with* $|F_i| \geq 3$;
(5) $|Min(R)| = 2$ *and each minimal prime ideal of* R *has at least three distinct elements;*
(6) $\Gamma(R) = K_{m,n}$ *with* $m, n \geq 2$;
(7) $AGC(R) = K_{m,n}$ *with* $m, n \geq 2$.

Proof. (1) $\Rightarrow$ (2). Since $gr(AGC(R)) = 4$. $AGC(R) = \Gamma(R)$ by Theorem 0.2.7, and thus $gr(\Gamma(R)) = 4$. (2) $\Rightarrow$ (3). Assume that $AGC(R) = \Gamma(R)$ and $gr(\Gamma(R)) = 4$. Thus $gr(\Gamma(R)) = 4$. (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) are clear by Theorem 0.3.9. (6) $\Rightarrow$ (7). Since (6) implies $|Min(R)| = 2$ by Theorem 0.3.9, we conclude that $AGC(R) = \Gamma(R)$ by Theorem 0.3.8, and thus $gr(AGC(R)) = \Gamma(R) = 4$. (7) $\Rightarrow$ (1). Since $AGC(R)$ is a complete bipartite and $m, n \geq 2$. Clearly $gr(AGC(R)) = 4$ □

Theorem 0.3.11. *[2, Theorem 2.4] The following statement are equivalent for a reduced commutative ring R.*
(1) $gr(\Gamma(R)) = \infty$.
(2) $T(R) = \mathbb{Z}_2 \times F$*, where each* F *is a field .*
(3) $\Gamma(R) = K_{1,n}$ *for some* $n \geq 1$.

Theorem 0.3.12. *Let R be a reduced commutative ring. Then the following statements are equivalent:*
(1) $gr(AGC(R)) = \infty$*;*
(2) $AGC(R) = \Gamma(R)$ *and* $gr(\Gamma(R)) = \infty$*;*
(3) $gr(\Gamma(R)) = \infty$*;*
(4) $T(R)$ *is ring -isomorphic to* $\mathbb{Z}_2 \times F$*, where each* F *is a field;*
(5) $|Min(R)| = 2$ *and at least one minimal prime ideal of R has exactly two distinct elements;*
(6) $\Gamma(R) = K_{1,n}$ *with* $n \geq 1$*;*
(7) $AGC(R) = K_{1,n}$ *with* $n \geq 1$.

Proof. (1) $\Rightarrow$ (2). Since $gr(AGC(R) = \infty$, $AGC(R) = \Gamma(R)$ by Theorem 0.2.7, and thus $gr(AGC(R)) = \infty$. (2) $\Rightarrow$ (3). Given, $AGC(R) = \Gamma(R)$ and $gr(\Gamma(R)) = \infty$. Thus $gr(\Gamma(R)) = \infty$. (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) are clear by Theorem 0.3.11. (6) $\Rightarrow$ (7). Since (6) implies $|Min(R) = 2|$ by Theorem 0.3.11, we conclude that $AGC(R) = \Gamma(R)$ by Theorem 0.3.8 and thus $gr(AGC(R)) = gr(\Gamma(R)) = \infty$. (7) $\Rightarrow$ (1) Since $AGC(R)$ is a star graph. Thus $gr(AGC(R)) = \infty$. □

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