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OSCILLATION BEHAVIOR OF SOLUTIONS OF SECOND ORDER DELAY DIFFERENCE EQUATIONS

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Abstract

This article, concerned with oscillation behavior of solutions of second order delay difference equations .We present new oscillation results by utilization the Riccati substitution technique and give an example to illustrate the main result.

Keywords: Second-order,oscillatory Solutions,delay difference equations.

INTRODUCTION

The second order difference equation have been the subject of investigations by many authors; See for examples [1-4,6,8-12]. In this paper,we will be concerned with a oscillatory behavior of solution of second order delay difference equations of the form.

$$\Delta\,(r_l\Delta(z_l)) + k_l\, g(z_{l-\tau}) = 0 \quad (1.1)$$

Where, {r_l} and {k_l }is an eventually positive real sequences; and g is sequence of reals and non-decreasing with zgz ⩾ 0 ; for z ≠ 0

and $\frac{g(z)}{z} \geq \mu$; μ > (1.2)

and $\lim_{z\to\infty} \frac{z}{g(z)}$ = N < ∞ , N>0.(1.3)

By a solution of Eq. (1.1), we mean a real sequence { z_l} satisfies Eq. (1.1) for

l⩾l0 . we consider only those solution z_l of Eq. (1.1),which one nontrivial such a solution is said to be oscillatory if it has arbitrarily large zero and non oscillatory otherwise.

2.Preliminary Lemmas

In the following section, we present the auxiliary lemmas which help in validating the important of results.

Lemma 2.1:[5] Let γ ⩾ 1 be a ratio of two odd numbers P,R,A,B ∈ R . Then

$$P^{\frac{\gamma+1}{\gamma}} - (P-R)^{\frac{\gamma+1}{\gamma}} \leq \frac{R^{\frac{1}{\gamma}}}{\gamma}\left[(1+\gamma)P - R\right],\ P, R \geq 0.$$

and

$$A_x - B_x^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \cdot \frac{A^{\gamma+1}}{B^\gamma}, \quad B > 0 \cdot$$

Lemma 2.2:[13] (Discrete Kneser Theorem). Let {z_l} be defined for l ⩾l_0 , and z_l>0 with Δ^m z_l of constant sign for l ⩾l_0 .And not

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identically zero .Then,there exists an integer h, 0⩽j⩽m , with (m+h) and odd for Δ^m z_l⩽0 and (m+h) even for Δ^m z_l⩾0 such that,

i. h⩽ m-1 implies 〖(-1)〗^(h+g) Δ^g z_l >0 for all l ⩾l_0 , h⩽g⩽m-1 and
ii. h⩾1 implies Δ^g z_l>0, for all large l⩾ l_0, l⩽g⩽h-1.

Lemma 2.3: [13] Let {z_l} be defined for l⩾l_0 and z_l>0 with Δ^m z_l ⩽0 for l ⩾l_0 and not identically zero .Then there exists a large l_1⩾ l_0 such that

$$z_l \geq \frac{(l-l_1)^{m-1}}{(m-1)!}\Delta^{m-1} z_{2^{m-h-1}n} , \quad l \geq l_1 .$$

Where l is defined as in lemma 2.2, further if {z_l} is increasing , then .

$$z_l \geq \frac{1}{(m-1)!}\left(\frac{1}{2^{m-1}}\right)^{m-1}\Delta^{m-1} \ z_l \ \text{ for all } l \geq 2^{m-1} l_1$$

Lemma 2.4:[7] Let {z_l} be an eventually Positive solution of Eq:(1.1).Then one of the following three cases holds for all sufficiently large l.

(F1) : z_l>0,(r_l Δ(z_l)>0,Δ (r_l Δ(z_l))⩽0;
(F2) :z_l>0,(r_l Δ(z_l)<0,Δ (r_l Δ(z_l))⩽0;
(F3) : z_l<0,(r_l Δ(z_l)>0,Δ (r_l Δ(z_l))⩽0;

Proof:The proof is similar to that of Lemma(2.1) of [7] and hence the details are omitted

3. Main Results.

In this segment, we will build up some wavering standards for Eq. (1.1)
For comfort , we signify

$$G_l = \sum_{s=l}^{\infty}\frac{1}{r_s} \quad , \quad S_l = \mu K_l \frac{(l-\text{c})^2}{l^2} ,$$
$$Q_l = \rho_l\,[S_l + \frac{1}{r_l {G_l}^2}] \quad , \quad \phi_l = \frac{\Delta\rho_l}{\rho_l} + \frac{2}{r_l G_l} .$$

and Δρ_l=max{0,Δρ_l}

In what follows , all happening utilitarian imbalances are accepted to hold ultimately, that is , they are fulfilled for all sufficiently enormous , As regular and without loss of over simplification , we can manage at least certain arrangements of equation.

Theorem 3.1:
Let the Eq. (1.3) holds . If

$$\lim_{S\to\infty} sup \sum_{s=l-\tau}^{l} G_s K_s > \mathbf{N} \ \ldots\ldots(3.1)$$

Then every solution of Eq. (1.1) is oscillatory.
Proof:
Accept in actuality , that {z_l} is a ultimately positive sequence, certain arrangement of Eq.(1.1). Eq. (3.1) suggests,

$$\sum_{s=l_0}^{l} G_s K_s < \infty .$$

There by z_l satisfies (F_2) of Lemma (2.4) , which yields

$$0 \leq z_{l-\tau} + r_l \,\Delta\, z_{l-\tau} \leq z_{l-\tau} \quad (3.2)$$

Setting;

$$\delta_l = z_l + r_l\,\Delta\, z_l\, G_l \,.\ (3.3)$$

We see that

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$\delta_l \leq z_l$, $g(\delta_{l-\tau}) \leq g(z_{l-\tau})$.

A straightforward calculation guarantees that Eq. (1.1) can be changed into the structure.

$\Delta(r_l \Delta z_l G_l + z_l) + G_l k_l g(z_{l-\tau}) = 0$,

which talking into account Eq. (3.5) infers that δ_l is a positive dignify arrangement of first order delay difference inequality.

$\Delta\delta_l + G_l k_l g(\delta_{l-\tau}) \leq 0$

Summing from l-τ to l-1 we get

$$\delta_{l-\tau} \geq \sum_{s=l-\tau}^{l-1} G_s k_s g(\delta_{l-\tau}) \frac{\delta_{l-\tau}}{g\,(\delta_{l-\tau})}$$
$$\geq g\,(\delta_{l-\tau}) \sum_{s=l-\tau}^{l-1} G_s k_s \,.$$

Thus we get,

$$\frac{\delta_{l-\tau}}{g\,(\delta_{l-\tau})} \geq \sum_{s=l-\tau}^{l-1} G_s k_s \,.$$

Hence ,

$$\lim_{S\to\infty} sup \sum_{s=l-\tau}^{l} G_s K_s \leq \lim_{S\to\infty} sup \left(\frac{\delta_{l-\tau}}{g\,(\delta_{l-\tau})}\right) \leq N \,.$$

but is a contradiction.

Theorem 3.2:

Assume Δz_l< 0 , (r_l Δz_l) < 0 . If there exists positive sequence

ρ such that

$$\sum_{s=l}^{\infty} \left(Q_s - \frac{\rho_s r_s .(\phi_s)^2}{4}\right) = \infty \quad (3.6)$$

Then every solution of Eq. (1.1) is oscillatory.

Proof :

Assume that (F_2) holds , By lemma (2.3) , we find

$z_l \geq (\frac{l}{2})\Delta z_l \quad \frac{\Delta z_l}{z_l} \leq \frac{2}{l}$

Summing from l-τ to l-1 we get ,

$$\sum_{s=l-\tau}^{l-1} \frac{\Delta z_l}{z_l} \leq \frac{2}{l} \; \frac{z_l}{z_{l-\tau}} \leq \frac{l^2}{(l-\tau)^2} \; \frac{z_{l-\tau}}{z_l} \geq \frac{(l-\tau)^2}{l^2} \quad (3.7)$$

It follows from (r_l Δz_l) ⩽ 0 ,we get

$$\Delta z_s \leq \left(\frac{r_l}{r_s}\right) \Delta z_l \,.$$

Summing from l to l_1 , we get

$$\sum_{s=l}^{l_1} \Delta z_s \leq \sum_{s=l}^{l_1} \frac{r_l}{r_s} \Delta z_l$$
$$z_{l_1} \leq z_l + r_l \Delta z_l \sum_{s=l_1}^{l} \frac{1}{r_s} \quad (3.8)$$

Letting l_1→ ∝ ,we obtain

$$z_l \geq G_l r_l \Delta z_l$$

Which implies that

$$\Delta \left(\frac{z_l}{G_l}\right) \geq 0 \,.$$

Define the function χ_l by

$$\chi_l = \rho_l \left[\frac{r_n \Delta z_l}{z_l} + \frac{1}{G_l}\right] \quad (3.9)$$

Then

$\chi_l > 0$ for $l \geq l_1$

$$\Delta \chi_l = \Delta\rho_l \left[\frac{r_l \Delta Z_l}{Z_l} + \frac{1}{G_l}\right] + \rho_{l+1} \frac{\Delta(r_l \Delta Z_l)}{Z_l}$$

$$\Delta \chi_l = \Delta\rho_l \left[\frac{r_l \Delta z_l}{z_l} + \frac{1}{G_l}\right] + \rho_{l+1} \left[\frac{\Delta(r_l \Delta z_l)}{z_l} + r_{l+1}\Delta z_{l+1}\Delta\left(\frac{1}{z_l}\right)\right] + \rho_{l+1}\left(\frac{1}{G_{l+1}} - \frac{1}{G_l}\right).$$

$$\Delta \chi_l = \Delta\rho_l \left[\frac{r_l \Delta z_l}{z_l} + \frac{1}{G_l}\right] + \rho_{l+1} \left[\frac{z_l \Delta(r_l \Delta z_l) - r_l \Delta z_l \Delta z_l}{z_l z_{l+1}}\right] + \rho_{l+1}\left(\frac{1}{G_{l+1}} - \frac{1}{G_l}\right).$$

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$$\Delta \chi_l = \Delta\rho_l\left[\frac{r_l \Delta z_l}{z_l} + \frac{1}{G_l}\right] + \rho_{l+1}\left[\frac{\Delta(r_l \Delta z_l)}{z_{l+1}} - \frac{r_l\Delta^2 z_l}{z_l z_{l+1}}\right] + \rho_{l+1}\left(\frac{\Delta G_l}{G_{l+1}G_l}\right)$$

Using Eq.(3.9) we obtain

$$\Delta \chi_l = \Delta\rho_l\left[\frac{r_l \Delta z_l}{z_l} + \frac{1}{G_l}\right] + \rho_{l+1}\frac{\Delta(r_l \Delta z_l)}{z_l} - \rho_{l+1}r_l\left[\frac{\chi_l}{r_l\rho_l} - \frac{1}{r_l G_l}\right]^2 + \frac{\rho_{l+1}}{G_{l+1}G_l}\left(\frac{1}{r_l}\right)$$

$$\Delta \chi_l = \frac{\Delta\rho_l}{\rho_l}\left[\rho_l\frac{r_l \Delta z_l}{z_l} + \frac{1}{G_l}\right] + \rho_{l+1}\frac{\Delta(r_l \Delta z_l)}{z_l} - \rho_{l+1}r_l\left[\frac{\chi_l}{r_l\rho_l} - \frac{1}{r_l G_l}\right]^2 + \frac{\rho_{l+1}}{G_{l+1}G_l}\left(\frac{1}{r_l}\right).$$

$$\Delta \chi_l = \frac{\Delta\rho_l}{\rho_l}\chi_l + \rho_{l+1}\frac{\Delta(r_l \Delta z_l)}{z_l} - \rho_{l+1}r_l\left[\frac{\chi_l}{r_l\rho_l} - \frac{1}{r_l G_l}\right]^2 + \frac{\rho_{l+1}}{G_{l+1}G_l}\left(\frac{1}{r_l}\right). \quad (3.10)$$

Using lemma (2.1) with
$P=\frac{\chi_l}{r_l\rho_l}$; $R=\frac{1}{r_l G_l}$; $\vartheta = 1$

We get

$$\left[\frac{\chi_l}{r_l\rho_l} - \frac{1}{r_l G_l}\right]^2 \geq \left(\frac{\chi_l}{\rho_l r_l}\right)^2 - \left(\frac{1}{r_l G_l}\right)\left[\frac{2\chi_l}{\rho_l r_l} - \frac{1}{r_l G_l}\right] \quad (3.11)$$

From (3.10) and (3.11) we obtain ;

$$\Delta \chi_l \leq \frac{\Delta\rho_l}{\rho_l}\chi_l + \rho_{l+1}\frac{\Delta(r_l \Delta z_l)}{z_l} - \rho_{l+1}r_l\left[\left(\frac{\chi_l}{\rho_l r_l}\right)^2 - \left(\frac{1}{r_l G_l}\right)\left[\frac{2\chi_l}{\rho_l r_l} - \frac{1}{r_l G_l}\right]\right]$$

$$\Delta \chi_l \leq \frac{\Delta\rho_l}{\rho_l}\chi_l + \rho_{l+1}\frac{\Delta(r_l \Delta z_l)}{z_l} - \rho_{l+1}r_l\left(\frac{\chi_l}{\rho_l r_l}\right)^2 + \rho_{l+1}r_l\left(\frac{1}{r_l G_l}\right)\left[\frac{2\chi_l}{\rho_l r_l} - \frac{1}{r_l G_l}\right]$$

$$\Delta \chi_l \leq \frac{\Delta\rho_l}{\rho_l}\chi_l + \rho_{l+1}k_l\mu\frac{z_{l-\tau}}{z_l} - \rho_{l+1}r_l\left(\frac{\chi_l}{\rho_l r_l}\right)^2 + \rho_{l+1}r_l\left(\frac{1}{r_l G_l}\right)\left[\frac{2\chi_l}{\rho_l r_l} - \frac{1}{r_l G_l}\right] \quad (3.12)$$

From Eq.(3.7) and Eq.(3.12)

$$\Delta \chi_l \leq \frac{\Delta\rho_l}{\rho_l}\chi_l + \rho_{l+1}k_n\mu\frac{(l-\tau)^2}{l^2} - \rho_{l+1}r_l\left(\frac{\chi_l}{\rho_l r_l}\right)^2 + \rho_{l+1}r_l\left(\frac{1}{r_l G_l}\right)\left[\frac{2\chi_l}{\rho_l r_l} - \frac{1}{r_l G_l}\right]$$

$$\Delta \chi_l \leq \frac{\Delta\rho_l}{\rho_l}\chi_l + \rho_{l+1}s_n - \rho_{l+1}r_l\left(\frac{\chi_l}{\rho_l r_l}\right)^2 + \rho_{l+1}r_l\left(\frac{1}{r_l G_l}\right)\left[\frac{2\chi_l}{\rho_l r_l} - \frac{1}{r_l G_l}\right]$$

This implies that

$$\Delta \chi_l \leq \left(\frac{\Delta\rho_l}{\rho_l} + \frac{2}{G_l r_l}\right)\chi_l - \left(\frac{1}{\rho_l r_l}\right)\chi_l^2 - \rho_{l+1}\left(s_n + \frac{1}{r_l G_l^2}\right).$$

Thus,by Eq.(3.10) yield

$$\Delta \chi_l \leq \phi_l\chi_l - \left(\frac{1}{\rho_l r_l}\right)\chi_l^2 - Q_l$$

$$\Delta \chi_l \leq -Q_l + \phi_l\chi_l - \left(\frac{1}{\rho_l r_l}\right)\chi_l^2$$

Applying the Lemma(2.1) With

$A=\phi_l$,$B=\left(\frac{1}{\rho_l r_l}\right)$ and $x=\chi_l$,we get

$$\Delta \chi_l \leq \left(-Q_l + \frac{\rho_l r_l \phi_l^2}{4}\right) \quad (3.13)$$

Summing from l to l_1 we get

$$\sum_l^{l_1}\left(Q_l - \frac{\rho_l r_l \phi_l^2}{4}\right) \leq \chi_{l_1} - \chi_l \leq \chi_l$$

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Which contradicts Eq.(3.6).The proof is complete.

Example 3.1: Consider the second order delay difference equation.

$\Delta(l^2(\Delta 2\ l))+2(2l^2+2\ l+1)\tau_{l-3}=0\quad l \geq 4$ (3.14)

Here: $r_l= l^2$, $K_l=2(2l^2+2\ l+1)$ and $\tau = 3$
One can show that all the important conditions of Theorem(3.1) are satisfied .]

We have $G_l = \sum_{s=l}^{\infty} \frac{1}{s^2} < \infty$

Set μ=1
$\lim_{l\to\infty} sup \sum_{s=l}^{\infty} G_s K_s = \lim_{l\to\infty} sup \sum_{s=l}^{\infty} \frac{2s^2+2s+1}{s^2} < \infty$.

Hence by Theorem (3.1),every solution of Eq.(3.1) is oscillatory

Example 3.2: Consider the second order delay difference equation.

$\Delta(2(\Delta z_l))+16z_{l-1}=0,\ l\geq 2$ (3.15)

Here: $r_l=2$, $K_l=16$ and $\tau = 1$

One can show that all the important conditions of Theorem(3.1) are satisfied .

Set $\mu = 1$ and $\rho_l=l$

we obtain
$G_l = \sum_l^{\infty} \frac{1}{2} < \infty$,and $\sum_{s=l}^{\infty} \left(Q_s - \frac{\rho_s r_s \phi_s^{\,2}}{4}\right)=\infty$.

Hence by Theorem (3.2),every solution of Eq.(3.2) is oscillatory, one such solution is $z_l = \frac{(-1)^l}{2}$.

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