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(U, V) - LUCAS POLYNOMIAL AND BI-UNIVALENT FUNCTION

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Abstract

In this paper, by using Lucas polynomials, developed a new family of bi-univalent functions D_Σ (s,t,z). Also, obtained (U, V) - Lucas polynomial, coefficient estimates and Fekete - Szegö inequalities for this new class D_Σ (s,t,z). Mathematics Subject Classification 2010: 30C45.

Keywords: Analytic function, Bi-univalent function, Coefficient estimate, Lucas polynomial, Sakaguchi type function.

Introduction

Let D be the unit disk $z: z \in C$ and $|z| < 1$, be the class of all functions analytic, satisfying the conditions $f(0) = 0$ and $f'(0) = 1$. Then each function f in $\mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad 1.1$$

Further, by S we shall denote the class of all functions in $\mathcal{A}$ which are univalent in D. By Koebe One-Quarter Theorem [7], the inverse of any univalent function $f(z) \in S$ can be satisfied as $f^{-1}(f(z)) = z(z \in D)$ and $f^{-1}(f(w)) = w(w \in D),\ |w| < r_0(l) \geq \frac{1}{4}$.

In general,

$$g(w) = w - a_2 w^2 + (2a^2{}_2 - a_3)w^3 - (5a^3{}_2 - 5a_2 a_3 + a_4)w^4$$
$$= w + \sum_{\eta=2}^{\infty} A_n w^n \qquad 1.2$$

Historically Lewin [12], investigated the class of bi-univalent functions and obtained a bound $|a_2| \leq 1.51$. Motivated by the work of Lewin [12], Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$. If function $f(z)$ and $f^{-1}(z)$ are univalent in D then a function $f(z) \in A$ is bi – univalent in D. Let Σ be indicating the class of all bi-univalent functions. We denote this subordination by $f \prec g\ (or)\ f(z) \prec g(z)$.

In particular, if the function g is univalent in D, the above subordination is equivalent to $f(0) = g(0), f(D) \subset g(D)$.

The coefficient estimate problem for $|a_n|, [(n \in N), n \geq 3]$ is still open [13]. Brannan and Taha [6] also worked on certain subclasses of the bi-univalent function class A and obtained estimates for their initial co-efficient. Various classes of bi-univalent functions were introduced and studied in recent times; the study of bi-univalent functions gained momentum mainly due to the work of Srivastava et al [13]. Motivates by this, many researchers [1], [3, 4], [19, 10], [13],

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[14] and [16, 17], also the references cited there in recently investigated several interesting subclasses of the class and found non-sharp estimates on the first two Taylor-Maclaurin co-efficients.

Recently, many researchers have been exploring bi-univalent functions, few to mention Fibonacci polynomials, Luca's polynomials, Chebyshev polynomials, Pell polynomials, Lucas-Lehmer polynomials, Orthogonal polynomials and the other special polynomials and their generalizations are of great importance in a variety of branches such as Physics, Engineering, Architecture, Nature, Art, Number theory, Combinatorics and Numerical analysis. These polynomials have been studied in several papers from a theoretical point view (see for example, [15, 18]).

Recall the following results relevant for our study as stated in Altinkaya and Yalçin[2]. Let u(x) and v(x) be polynomials with real coefficients. The (u,v)-Lucas polynomials $L_{U,V,t}(x)$ are defined by the recurrence relation

$$L_{U,V,t}(x) = u(x)L_{U,V,t-1}(x) + v(x)L_{U,V,t-2}(x) \quad t \geq 2$$

From which the first few Lucas polynomials can be found as

$$\begin{aligned} L_{U,V,0}(x) &= 2 \\ L_{U,V,1}(x) &= u(x) \\ L_{U,V,2}(x) &= u^2(x) + 2v(x) \\ L_{U,V,3}(x) &= u^3(x) + 3u(x)v(x), \ldots \end{aligned} \qquad 1.3$$

For the special cases of u(x) and v(x), we can get the polynomials given $L_{x,1,0}(x) = L_n(x)$ Lucas Polynomials, $L_{2x,1,0}(x) = D_n(x)$, Pell-Lucas polynomials, $L_{1,2x,n}(x) = j_n(x)$, Jacobsthal-Lucas polynomials, $L_{3x,-2n,n}(x) = F_n(x)$) Fermate-Lucas polynomials, $L_{2x,-1,n}(x) = T_n(x)$, Chebyshev polynomials first kind.

Lemma 1.1. [11] Let $G_{\{L(x)\}}(z)$ be the generating function of the (u,v)- Lucas polynomial sequence $L_{U,V,t}(x)$.

Then

$$G_{\{L(x)\}}(z) = \sum_{n=0}^{\infty} L_{U,V,t}(x)z^n = \frac{2 - u(x)z}{1 - u(x)z - v(x)z^2}$$

$$G_{\{L(x)\}}(z) = G_{\{L(x)\}}(z) - 1 = 1 + \sum_{n=0}^{\infty} L_{U,V,t}(x)z^n$$

$$= \frac{1 + v(x)z^2}{1 - u(x)z - v(x)z^2}$$

Definition 1.1

For $\lambda \geq 1, |s| \leq 1, |t| \leq 1$ but $s \neq t$, a function $f \in A$ is called in the family $D_\Sigma(s,t,z)$ if the following subordinations are satisfied:

$$\frac{(s-t)z(f'(z))^\lambda}{f(sz) - f(tz)} \prec Ꝁ\{L_{U,V,t}(x)\}(z) - 1 \qquad 1.4$$

$$\frac{(s-t)\omega(g'(z))^\lambda}{g(s\omega) - g(t\omega)} \prec Ꝁ\{L_{U,V,t}(x)\}(\omega) - 1 \qquad 1.5$$

where $Ꝁ_{\{L(u,v,t)\}}(z) \in \emptyset$ and the function is $g(w) = f^{-1}(w)$.

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2 Coefficient Estimates for the class $D_\Sigma(s,t,z)$

Theorem 2.1 Let $f(z) \in D_\Sigma(s,t,z)$, for $\lambda \geq 1, |s| \leq 1, |t| \leq 1$ but $s \neq t$,

Then

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{|U^2(x)2(\lambda^2+2\lambda)-2(2\lambda-s-t)(U^2(x)+2V(x)|}}$$

$$|n_3| \leq \frac{U^2(x)}{[2\lambda-s-t]^2}+\frac{|U(x)|}{6\lambda}$$

Proof.

Let $f(z) \in D(s,t,z)$. then, according to the Definition (2.1), for some holomorphic functions c, d such that $d(0) = c(0) = 0, |d(\omega)| < 1, |c(z)| < 1, (z,\omega \in D)$, then write

$$\frac{(s-t)z(f'(z))^\lambda}{f(sz)-f(tz)} = Ҕ\{L_{U,V,t}(x)\}(c(z))-1 \qquad 2.1$$

$$\frac{(s-t)\omega(g'(z))^\lambda}{g(s\omega)-g(t\omega)} = Ҕ\{L_{U,V,t}(x)\}(d(\omega))-1 \qquad 2.2$$

For z, ω ∈ U, it known before that if $|c(z)| = \left|\sum_{j=1}^{\infty} y_j z^j\right| < 1$ and $|d(\omega)| = \left|\sum_{j=1}^{\infty} \mu_j \omega^j\right| < 1$,

Then $|y_j| < 1$ and $|\mu_j| < 1$ where $j \in N = \{1,2,3,\dots\}$. by equivalence

$$\frac{(s-t)z(f'(z))^\lambda}{f(sz)-f(tz)} = -1+L_{U,V,0}(x)+L_{U,V,1}(x)c(z)+L_{U,V,2}(x)c^2(z)+\cdots \qquad 2.3$$

$$\frac{(s-t)\omega(g'(z))^\lambda}{g(s\omega)-g(t\omega)} = -1+L_{U,V,0}(x)+L_{U,V,1}(x)d(\omega)+L_{U,V,2}(x)d^2(\omega)+\cdots \qquad 2.4$$

From the equalities (2.1) and (2.2), yields

$$\frac{(s-t)z(f'(z))^\lambda}{f(sz)-f(tz)} = -1+L_{U,V,0}(x)y_1z+[L_{U,V,1}(x)y_2+L_{U,V,2}(x)y_1{}^2]z^2+\cdots \qquad 2.5$$

$$\frac{(s-t)\omega(g'(z))^\lambda}{g(s\omega)-g(t\omega)} = -1+L_{U,V,0}(x)y_1w+[L_{U,V,1}(x)y_2+L_{U,V,2}(x)y_1{}^2]\omega^2+\cdots \qquad 2.6$$

It follows that, from (2.5) and (2.6) obtain

$$(2\lambda-s-t)n_2 = L_{U,V,0}(x)y_1, \qquad 2.7$$

$$\lambda n_3+2\lambda(\lambda-1)n_2^2 = L_{U,V,1}(x)y_1+L_{U,V,1}(x)y_1^2 \qquad 2.8$$

$$-(2\lambda-s-t)n_2 = L_{U,V,1}(x)\mu_1, \qquad 2.9$$

$$3\lambda(2n_2^2-n_3)+2\lambda(\lambda-1)n_2^2 = L_{U,V,1}(x)\mu_2+L_{U,V,2}(x)\mu_1^2. \qquad 2.10$$

From (2.7) and (2.9)

$$y_1 = -\mu \qquad 2.11$$

$$2[(2\lambda-s-t)^2n_2^2] = L^2{}_{U,V,1}(x)(y_1^2+\mu_1^2). \qquad 2.12$$

Summation of (2.8) and (2.10) give us

$$2(3\lambda)2n_2^2+4\lambda(\lambda-1)n_2^2 = L_{U,V,1}(x)(y_2+\mu_2)+L_{U,V,1}(x)(y_1^2+\mu_1^2). \qquad 2.13$$

Hence, (2.12) minus (2.13) gives us

$$3\lambda(-2n_2^2-2n_3) = L_{U,V,1}(x)(y_2-\mu_2) \qquad 2.14$$

Then, by using (1.33) and (2.14) get

$$n_3 = n_2^2+\frac{L_{U,V,1}(x)(y_2-\mu_2)}{2(3\lambda)} \qquad 2.15$$

Next, in order to find the bound on $|n_3|$, by subtracting (2.10) from (2.8) obtain that

$$n_3 = \frac{L^2{}_{U,V,1}(x)(y_1^2+\mu_1^2)}{2[2\lambda-s-t]^2}+\frac{L_{U,V,1}(x)(y_2-\mu_2)}{2(3\lambda)}$$

$$|n_3| \leq \frac{U^2(x)}{[2\lambda-s-t)]^2}+\frac{|U(x)|}{6\lambda}$$

Thus, the proof of our main theorem is completed.

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3 Feketo Szegö inequality for the class $D_\Sigma(s,t,z)$

Theorem 3.1. [8] Let the function $f \in \Sigma'$ given by equation (1.1) be in the class $D_\Sigma(s,t,z)$, for $\lambda \geq 1$, $|s| \leq 1$, $|t| \leq 1$ but $s \neq t$, and for some $\mu \in R$, Then

$$|n_3 - \mu n_2^2| \leq \begin{cases} \dfrac{h_2(x)}{2m^2e^{-m}n_3(3-s^2-t^2-st)} & 0 \leq h(\mu) \leq \dfrac{1}{m^2e^{-m}n_3(3-s^2-t^2-st)} \\ h_2(x)h(\mu) & h(\mu) \geq \dfrac{1}{m^2e^{-m}n_3(3-s^2-t^2-st)} \end{cases}$$

$$h(\mu) = \frac{(1-\mu)h^2{}_2(x)}{2(2s+2t-s^2-t^2-2st)h^2{}_2(x)me^{-m} - 2[(2-s-t)^2m^2e^{-2m}]h_3(x)}$$

Proof.

From the equation (2.16) and (2.17) get

$$n_3 - \mu n_2^2 = \left[\frac{h_2(x)\,(p_2 - q_2)}{2m^2e^{-m}n_3(3-s^2-t^2-st)}\right] + \frac{(1-\mu)h_2(x)\,(p_2+q_2)}{2(2s+2t-s^2-t^2-2st)me^{-m} - \dfrac{2[(2-s-t)^2m^2e^{-2m}]h_3(x)}{h^2{}_2(x)}}$$

$$n_3 - \mu n_2^2 = h_2(x)\left\{\left[h(\mu) + \frac{1}{2m^2e^{-m}n_3(3-s^2-t^2-st)}\right]y_2 + \left[h(\mu) - \frac{1}{2m^2e^{-m}n_3(3-s^2-t^2-st)}\right]\mu_2\right\}$$

where

where

$$h(\mu) = \frac{(1-\mu)h^2{}_2(x)}{2(2s+2t-s^2-t^2-2st)h^2{}_2(x)me^{-m} - 2[(2-s-t)^2m^2e^{-2m}]h_3(x)}$$

Then view of equation

$$|n_3 - \mu n_2^2| \leq \begin{cases} \dfrac{h_2(x)}{2m^2e^{-m}n_3(3-s^2-t^2-st)} & 0 \leq h(\mu) \leq \dfrac{1}{m^2e^{-m}n_3(3-s^2-t^2-st)} \\ h_2(x)h(\mu) & h(\mu) \geq \dfrac{1}{m^2e^{-m}n_3(3-s^2-t^2-st)} \end{cases}$$

Taking $\mu = 1$ get

Corollary 3.2

Let the function $f \in \Sigma'$ given by (1.1) be in the class $s_\Sigma(g,s,t)$ $s,t \in \mathbb{C}$ with $s \neq t$ and $|s| \leq 1, |t| \leq 1$. Then

$$|n_3 - n_2^2| \leq \frac{h_2(x)}{2m^2e^{-m}n_3(3-s^2-t^2-st)}$$

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