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Oscillatory Behavior of Third Order Neutral Difference Equations with Distributed Delay and Damping Term

G.Sumitha
PG & Research,
Department of Mathematics,
Kandaswami kandar's college,
Namakkal, Tamil Nadu, India.

V.Ramesh
PG & Research,
Department of Mathematics,
Kandaswami kandar's college,
Namakkal, Tamil Nadu, India.

Abstract

In the present paper, we investigate the oscillatory behavior of third-order neutral difference equation with damping and distributed delay. The equation is given as follows.

$$\Delta\left(r_m\Delta\left(\beta_m\Delta\left(x_m+\sum_{m=a}^{b}p_m x_{m-\mu}\right)\right)\right)+n_m\Delta\left(\beta_m\Delta\left(x_m+\sum_{m=a}^{b}p_m x_{m-\mu}\right)\right)+G(x_{m-\tau})=0$$

Keywords: third – order, oscillatory solutions, delay and damping term. 2010 MSC: 39A10.

1. Introduction

In the present paper, we investigate the oscillatory behavior of third-order neutral difference equation with damping and distributed delay. The equation is given as follows

$$\Delta\left(r_m\Delta\left(\beta_m\Delta\left(x_m+\sum_{m=a}^{b}p_m x_{m-\mu}\right)\right)\right)+n_m\Delta\left(\beta_m\Delta\left(x_m+\sum_{m=a}^{b}p_m x_{m-\mu}\right)\right)+G(x_{m-\tau})=0$$

Throughout this article, we always make the hypotheses as follows

- (H_1): $\{r_m\}\{n_m\}$ are positive real sequences with $\sum_{s=m_0}^{m-1}\frac{1}{r_s}exp\left(\sum_{s=m_0}^{m-1}\frac{n_s}{r_s}\right)=\infty$
- (H_2): $\{\beta_m\}$ is real sequence with $\sum_{m=m_0}^{\infty}\frac{1}{\beta_m}=0$
- (H_3): $\{\rho_m\}$ is sequence with $0\leq\sum_{u=a}^{b}\rho_u\leq\rho\leq1$
- (H_4): μ and τ are positive integers,
- (H_5) :G be a function $\frac{G(\omega)}{\omega}\geq q_\omega$.

Letting

$$y_m=x_m+\sum_{m=a}^{b}p_m x_{m-\mu}$$

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A function x_m is a solution of equation (1.1) on positive real sequence for every $m-1\geq m_0$.with $x_m\beta_m y_m$ and $r_m\Delta(\beta_m\Delta y_m)$ be a positive real sequence. We focus on the solutions satisfying $sup\{|x_m|:m\leq m_1<\infty\}>0.m_1\geq m_0$. The solution with arbitrarily large zeros on positive sequence is treated as an oscillatory solution.

2. Preliminaries:

For the oscillatory solution of (1.1), we usually talk about the eventually positive solutions. In this section, the following, results may play an important role in establishing new oscillation criteria for (1.1).

Lemma 2.1: Assume that x_m is positive real sequences that there are two cases as follows,

i) $y_m>0,\Delta y_m>0,\ \Delta(\beta_m\Delta y_m)>0,$
ii) $y_m>0,\Delta y_m<0,\ \Delta(\beta_m\Delta y_m)>0,$

For $m\geq m_1\geq m_0$ with sufficiently large m_1.

Proof: We see that $\{x_m\}$ is the positive solution (1.1). Then if follows from (H_4). That $x_{m-\mu}>0$ and $x_{m-\iota}>0$ for $m_1\geq m_0$. with sufficiently large m_1. Respectively it is easy to get $y_m>x_m>0$

From (1.1) and (H_5) we get

$$\Delta\left(r_m\Delta(\beta_m\Delta(y_m))\right)+n_m\Delta(\beta_m\Delta(y_m))+G(x_{m-\tau})=0$$

$$\Delta\left(r_m\Delta(\beta_m\Delta(y_m))\right)+n_m\Delta(\beta_m\Delta(y_m))\leq-\sum_{m=c}^{d}G(x_{m-\tau})$$

$$\leq-\sum_{m=c}^{d}q(x_{m-\tau})$$
$$\leq 0$$

It follows that,

$$\sum_{s=m_0}^{m-1}\frac{1}{r_m}\exp\left(\sum_{s=m_1}^{m_0}\frac{n_s}{r_s}\right)=\infty.$$

$$\exp\left(\sum_{s=m_1}^{m_0}\frac{n_s}{r_s}\right)r_m\Delta(\beta_m\Delta(y_m))<0$$

Then

$\exp\left(\sum_{s=m_1}^{m_0}\frac{n_s}{r_s}\right)r_m\Delta(\beta_m\Delta(y_m))$ is a decreasing functions,with one sign eventually. Thus, from (H_1)

$\Delta(\beta_m\Delta(y_m))<0$ (or) $\Delta(\beta_m\Delta(y_m))>0$

for $m_2\geq m_1\geq m_0$

We claim that $\Delta(\beta_m\Delta(y_m))>0$, Suppose $\Delta(\beta_m\Delta(y_m))\geq 0$,

According to the monotonic of

$$\exp\left(\sum_{s=m_1}^{m_0}\frac{n_s}{r_s}\right)r_m\Delta(\beta_m\Delta(y_m))\leq -M$$

For M>0, summing the above inequality on $[m_2, m-1]$ to get

$$\Delta(\beta_m\Delta(y_m))\leq -M\sum_{s=m_2}^{m-1}\frac{1}{r_m}\exp\left(-\sum_{s=m_1}^{m-1}\frac{n_s}{r_s}\right)$$

$$\beta_m\Delta(y_m)\leq\beta_m\Delta(y_{m2})-M\sum_{s=m_2}^{m-1}\frac{1}{r_m}\exp\left(-\sum_{s=m_1}^{m-1}\frac{n_s}{r_s}\right)$$

Letting $m\to-\infty$. We have $\beta_m\Delta(y_m)\to-\infty$, because see (H_1). Than it follows from $\Delta(\beta_m\Delta(y_m))\leq 0$, that $\beta_m\Delta(y_m)\leq\beta_{m3}\Delta(y_{m3})$ for

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$m \geq m_3 \geq m_2$. Dividing by β_m and summing on $[m_3, m]$, we have that

$$y_m - y_{m3} \leq \beta_{m3} \Delta(y_{m3}) \sum_{s=m_3}^{m-1} \frac{1}{\beta_s}.$$

From condition (H_5), we have $y_m \to -\infty$ as $m \to \infty$. this contradicts $y_m > 0$, which implies $\Delta(\beta_m \Delta(y_m)) > 0$. We complete the proof.

Lemma 2.2: Assume that x_m is the positive solution of (1.1) and y_m satisfied case (II). Suppose

$$\sum_{m=m_1}^{\infty} \frac{1}{\beta_v} \sum_{m=v}^{\infty} \frac{1}{r_u} \sum_{m=u}^{\infty} q_m = \infty, \quad (2.1)$$

Where $q_m = \sum_{m=c}^{d} q_\omega$. then $\lim_{m\to\infty} x_m = 0$

Proof: Suppose, we see that x_m is the positive solution of (1.1) for (m_0, ∞) Due to the fact that case (II) is valid for y_m , we obtain $\lim_{m\to\infty} y_m = l > 0$. and that we use proof by contradiction to prove $l = 0$. Suppose $l > 0$. Then it follows that $l + \epsilon > y_m > l$ for $\epsilon > 0$with $m \geq m_1 \geq m_0$.Taking ϵ such that $p\epsilon < l(1-p)$, from (H_3), (H_4) , and property (II), we have

$$y_m = x_m + \sum_{m=a}^{b} p_m x_{m-\mu}$$
$$x_m = y_m - \sum_{m=a}^{b} p_m y_{m-\mu}$$
$$\left(\Delta k_m r_m \Delta(\beta_m \Delta(y_m))\right) \leq S k_m y_{m-\tau} q$$

Summing on $[m, \infty]$ to obtain

$$-k_m r_m \Delta(\beta_m \Delta(y_m)) + S \sum_{m=m_0}^{\infty} y_{m-\tau} q \leq 0.$$

By morality of $y_{m-\tau} > l$ and $\Delta k_m > 0$, we conclude

$$r_m \Delta(\beta_m \Delta(y_m)) + Sl \sum_{m=u}^{\infty} q_\omega = 0$$

This yields

$$\beta_m \Delta(y_m) + \frac{Sl}{r_u} \sum_{m=u}^{\infty} q_\omega < 0$$
$$\Delta(y_m) + \frac{Sl}{\beta_m} \sum_{m=v}^{\infty} \frac{1}{r_u} \sum_{m=u}^{\infty} q_\omega < 0$$

Further summing on $[m_1, \infty]$ to get

$$\sum_{m=m_0}^{\infty} \frac{1}{\beta_m} \sum_{m=v}^{\infty} \frac{1}{r_u} \sum_{m=u}^{\infty} q_m = \infty,$$

This contradicts (2.1), which leads to $l = 0$, and then $\lim_{m\to\infty} x_m = 0$ from $y_m > x_m > 0$. we completed the proof.

3. Oscillation result

Some new Oscillation criteria for (1.1) are obtained by using the generalized Riccati transformation, inequality estimation, integral averaging techniqye and Philos type technique

We define function h, $H: N_0 \times N_0 \to R$ such that there exists a double sequence $\{H_{m,n} : m \geq n \geq 0\}$ such that,

i) $H_{m,m} = 0 \; for \; m > 0$

ii) $H_{m,m} = 0 for \; m > n \geq 0$

iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} < 0 \; for \; m \geq n \geq 0.$

$$\Delta_2 H_{m,n} = -h(m,s)\sqrt{H(m,s)}$$

Theorem 3.1: Assume that (2.1) hold and there exist $H \in X$ and ξ be a positive sequence, such that

$$0 < \inf_{s \geq m_0} \left[\lim_{m\to\infty} inf \frac{H(m,s)}{H(m,m_0)}\right] \leq \infty \quad (3.1)$$

And

$$\sum_{s=m_0}^{\infty} \frac{\xi_m^{\,2}}{\rho_m r_m} = \infty \quad (3.2)$$

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$$\xi M \le \lim_{m\to\infty} sup \frac{1}{H(m,M)} \sum_{s=M}^{m} \left(H(m,s)p_s - \frac{S\rho_m r_m h^2(m,s)}{4} \right) \quad (3.3)$$

For $m \ge M \ge m_0, S > 1, \theta > 0$, where

$$R_m = \rho_m \frac{(1-p)\theta q_m}{\beta_m}, \quad \rho_m = exp \sum_{s=m_0}^{m} \frac{n_s}{r_s},$$
$$\xi_m = max \{\phi_m, 0\} \quad (3.4)$$

Then any solution x_m of (1.1) either oscillates or converges to zero.

Proof: Let x_m be a non-oscillatory solution of (1.1) without loss of generality, we assume that x_m >0 for (H_4) and (H_5), we have $x_{m-\mu} > 0$, for $n \ge n_1 \ge n_0 \in N_0$ for sufficiently large n_1. From Lemma (2.1), y_m is one case of (I) and (II).

Case (I):

Let y_m satisfies case (I) of Lemma (2.1) from (1.2), we have

$$y_m = x_m + \sum_{m=a}^{b} p_m x_{m-\mu}$$
$$\ge (1 - p_m) y_m$$
$$\ge (1-p) y_m$$

From (H_3) and (H_4). By (H_5) and the above inequality, it is obvious that

$$\Delta\left(r_m \Delta(\beta_m \Delta(y_m))\right) + n_m \Delta(\beta_m \Delta(y_m)) \le -(1-p) \sum_{m=c}^{d} q_\omega x_{m-\mu}$$
$$\le -(1-p) y_{m-\mu} q_\omega$$

Putting that $\mathcal{X}_m = \rho_m \frac{r_m \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m}$, $m \ge m_1$ with ρ_m given in (3.4) , we know

$$\Delta \mathcal{X}_m = \Delta\rho_m \frac{r_m \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m} + \rho_{m+1} \Delta \left[\frac{r_m \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m} \right]$$
$$\le \Delta\rho_m \frac{r_m \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m} + \rho_{m+1} \left[\frac{\Delta(r_m \Delta(\beta_m \Delta(y_m)))}{\beta_{m+1} \Delta y_{m+1}} \right] -$$
$$\rho_{m+1} \left[\frac{r_m \Delta(\beta_m \Delta(y_m)) \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m \, \beta_{m+1} \Delta y_{m+1}} \right]$$
$$\le \frac{\Delta\rho_m}{\rho_m} \mathcal{X}_m - \rho_{m+1} \left[\frac{(1-p) y_m - \mu q \omega}{\beta_{m+1} \Delta y_{m+1}} \right] - \rho_{m+1} \left[\frac{n_m (\beta_m \Delta(y_m))}{\beta_{m+1} \Delta y_{m+1}} \right] - \frac{\mathcal{X}_m^{\,2}}{\rho_{m+1} r_m}$$
$$\le -\rho_{m+1} \left[\frac{(1-p) y_m - \mu q \omega}{\beta_{m+1} \Delta y_{m+1}} \right] - \frac{\mathcal{X}_m^{\,2}}{\rho_{m+1} r_m}.$$

By property (I), there exist a limit of $\frac{1}{\Delta y_m}$ as $m \to \infty$, which is denoted by $\lim_{m\to\infty} \frac{1}{\Delta y_m} = \eta$. Choosing $\varepsilon = \frac{\eta}{2}$, we obtain $\frac{1}{\Delta y_m} > \frac{\eta}{2}$ for $m \ge m_2 \ge m_1$. Letting $\theta = \frac{y\delta\eta}{2}$, from $m - c \ge \delta$ in (H_4) we have

$$\Delta \mathcal{X}_m \le -R_{m+1} - \frac{\mathcal{X}_m^{\,2}}{\rho_{m+1} r_m}.$$

Where R_m is defined in (3.4). Multiply the above inequality by $H(m,s)$ and summing the inequality from m_2 to m, we get

$$\sum_{s=m_2}^{m} H(m,s) R_m \le \sum_{s=m_2}^{m} H(m,s) \mathcal{X}_m - \sum_{s=m_2}^{m} H(m,s) \frac{\mathcal{X}_m^{\,2}}{\rho_{m+1} r_m}$$
$$\sum_{s=m_2}^{m} H(m,s) R_m \le H(m,m_2) \mathcal{X}_m - \sum_{s=m_2}^{m} h(m,s) \sqrt{H(m,s)\, \mathcal{X}_m} - \sum_{s=m_2}^{m} H(m,s) \frac{\mathcal{X}_m^{\,2}}{\rho_{m+1} r_m}$$
$$\le H(m,m_2) \mathcal{X}_m - \sum_{s=m_2}^{m} \left(h(m,s) \sqrt{\frac{S\rho_m r_m}{2}} + \mathcal{X}_m \sqrt{\frac{H(m,s)}{S\rho_m r_m}} \right)^2 + \sum_{s=m_2}^{m} \frac{S\rho_m r_m h^2(m,s)}{4} -$$

From the integral averaging technique. Then

$$\limsup_{m\to\infty} \frac{1}{H(m,m_2)} \sum_{s=m_2}^{m} H(m,s) R_m - \frac{S\rho_m r_m h^2(m,s)}{4} \le$$
$$\mathcal{X}_{m2} - \lim_{m\to\infty} \frac{1}{H(m,m_2)} \sum_{s=m_2}^{m} \frac{(S-1) H(m,s) \mathcal{X}_m^{\,2}}{S\rho_m r_m}$$

Thus, it follows from (3.3) that

$$\xi_m \le \mathcal{X}_m \quad , \quad m \ge m_2,$$

And

$$\lim_{m\to\infty} inf \frac{1}{H(m,m_2)} \sum_{m=m_2}^{m} \frac{(S-1) H(m,s) \mathcal{X}_m^{\,2}}{S\rho_m r_m} \le \mathcal{X}_{m2} - \xi_{m2} < \infty \quad (3.5)$$

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Next we claim

$$\sum_{s=m_2}^{\infty} \frac{x_m^{\,2}}{\rho_m r_m} < \infty$$

Suppose $\sum_{s=m_2}^{\infty} \frac{x_m^{\,2}}{\rho_m r_m} = \infty$ it follows from (3.1) that

$$\inf_{S \geq m_0} \left[\lim_{m\to\infty} \frac{H(m,s)}{H(m,m_0)}\right] > \mu$$

For $\mu > 0$, and then $\frac{H(m,m_3)}{H(m,m_0)} > \mu$, for $m \geq m_3 \geq m_2$, thus we have

$$\sum_{s=m_3}^{\infty} \frac{x_m^{\,2}}{\rho_m r_m} \geq \frac{T_1}{\mu}$$

For $T_1 > 0$. then,
when $m \geq m_3$, we conclude

$$\frac{1}{H(m,m_0)} \sum_{m=m_3}^{m} \frac{H(m,s)x_m^{\,2}}{\rho_m r_m} = \frac{1}{H(m,m_0)} \sum_{s=m_3}^{m} - \frac{\partial H(m,s)}{\partial s} \sum_{s=m_3}^{s} \frac{x_m^{\,2}}{\rho_\eta r_\eta}$$
$$\geq \frac{1}{H(m,m_0)} \frac{T_1}{\mu} \sum_{s=m_3}^{m} - \frac{\partial H(m,s)}{\partial s}$$
$$\geq \frac{T_1}{\mu} \frac{H(m,m_3)}{H(m,m_0)}$$
$$\geq T_1.$$

This implies

$$\lim_{m\to\infty} \frac{1}{H(m,m_0)} \sum_{s=m_3}^{m} \frac{H(m,s)x_m^{\,2}}{\rho_m r_m} = \infty$$

This leads to a contradiction with (3.5), then we conclude $\lim_{m\to\infty} \frac{1}{H(m,m_0)} \sum_{s=m_3}^{m} \frac{H(m,s)x_m^{\,2}}{\rho_m r_m} < \infty$, which contradicts (3.2).

If y_m satisfies case (II), then $\lim_{m\to\infty} x_m = 0$, from (2.1) and Lemma (2.2). The proof is complete.

Theorem 3.2: Assume that (2.1) hold and there exist H$\in X$ and ω_m be a positive real sequence such that

$$\lim_{m\to\infty} \frac{1}{H(m,M)} \sum_{M}^{m} (H(m,s)B_s) - \frac{S_m r_m h^2(m,s)}{4(s-1)} \tag{3.6}$$

For $S > 1, \theta > 0$ and $m \geq M \geq m_0$,

Where

$$B_m = U_m - S\rho_m r_m B^2,$$
$$\rho_m = exp \sum_{s=m_0}^{m} \frac{n_s}{r_s} \tag{3.7}$$

$$U_m = \rho_m \left[r_m B_m^{\,2} - n_m B_m - \Delta(r_m B_m) + \frac{(1+R_m)\theta q_m}{\beta_m}\right] \tag{3.8}$$

Then any solution x_m of (1.1) either oscillates or converges to zero

Proof: Let x_m be a non-oscillatory solution of (1.1). Without loss of generality, we assert that x_m>0. On positive .It follows from (H_4) and (H_5) that x_m-μ>0, is positive real sequence for sufficiently large m_1 from Lemma (2.1), y_m is one case of (i) and (ii)

y_m Satisfies case (I), then by letting

$$x_m = \rho_m \left[\frac{r_m \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m} + r_m B_m\right], \qquad m \leq m_1,$$

We claim that

$$\Delta x_m = \Delta\rho_m \left[\frac{r_m \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m} + r_m B_m\right] + \rho_{m+1} \left[\frac{r_m \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m} + (r_m B_m)\right]$$

$$\Delta x_m = \Delta\rho_m \left[\frac{r_m \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m} + r_m B_m\right] + \rho_{m+1} \Delta \left[\frac{r_m \Delta(\beta_m \Delta(y_m))}{\beta_m \Delta y_m} + \Delta(r_m B_m)\right]$$

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$$\Delta X_m \le \Delta\rho_m\left[\frac{r_m\Delta(\beta_m\Delta(y_m))}{\beta_m\Delta y_m}+r_mB_m\right]+\rho_{m+1}\Delta\left[\frac{r_m\Delta(\beta_m\Delta(y_m))}{\beta_{m+1}\Delta y_{m+1}}\right]$$
$$-\rho_{m+1}\left[\frac{r_m\Delta(\beta_m\Delta(y_m))\Delta(\beta_m\Delta(y_m))}{\beta_m\Delta y_m\beta_{m+1}\Delta y_{m+1}}\right]+\rho_{m+1}\Delta(r_mB_m)$$

$$\Delta X_m \le \frac{\Delta\rho_m}{\rho_m}X_m-\rho_{m+1}\left[\frac{(1-R)y_{m-\mu}q_\omega}{\beta_{m+1}\Delta y_{m+1}}+\frac{n_m\Delta(\beta_m\Delta(y_m))}{\beta_{m+1}\Delta y_{m+1}}+\left(\frac{\Delta(\beta_m\Delta(y_m))}{\beta_{m+1}\Delta y_{m+1}}\right)^2\right]$$
$$+\rho_{m+1}\Delta(r_mB_m)$$

In the same way as Theorem (3.1) taking, $\lim_{m\to\infty}\frac{1}{\Delta y_m}=\eta.\ \xi=\frac{\eta}{2}$, and $\theta=\frac{y\delta\eta}{2}$, we have

$$\Delta X_m \le \frac{\Delta\rho_m}{\rho_m}X_m-\rho_{m+1}\left[\frac{(1-R)\theta q_m}{\beta_{m+1}}+\frac{n_mX_m}{\rho_{m+1}r_m}-n_mB_m+r_m\left(\frac{X_m}{\rho_{m+1}r_m}-B_m\right)^2\right]$$
$$+\rho_{m+1}\Delta(r_mB_m)$$

$$\Delta X_m \le -U_m+2B_mX_m-\frac{{X_m}^2}{\rho_{m+1}r_m}$$

$$\Delta X_m \le -U_m+2B_mX_m-\frac{{X_m}^2}{S\rho_{m+1}r_m}-\frac{(S-1){X_m}^2}{S\rho_{m+1}r_m}$$

Where U_m is defined by (3.8).

Based on $Bu-Au^2\le\frac{B^2}{4A}$ for A>0,u$\in\mathbb{R}$, we get

$$\Delta X_m \le -B_m-\frac{(S-1){X_m}^2}{S\rho_{m+1}r_m}$$

Where B_m is given by (3.7). Multiplying the inequality by $H(m,s)$ and integrating on $[M,m]$, We obtain

$$\sum_{s=m}^{m}H(m,s)B_m \le X_MH(m,M)-\sum_{s=M}^{m}\left(\left(h(m,s)-\sqrt{H(m,s)X_S}-\right.\right.$$
$$\left.\left.\frac{(S-1)H(m,s){X_m}^2}{S\rho_{m+1}r_m}\right)\right)^2$$

$$\le X_MH(m,M)-\sum_{s=M}^{m}\left(h(m,s)-\sqrt{\frac{S\rho_mr_m}{4(S-1)}}+X_m\sqrt{\frac{(S-1)H(m,s)}{S\rho_{m+1}r_m}}\right)^2+$$
$$\sum_{s=M}^{m}\frac{S\rho_mr_mh^2(m,s)}{4(S-1)}$$

Then

$$\frac{1}{H(m,M)}\sum_{s=M}^{m}(H(m,s)B_m-\frac{S\rho_mr_mh^2(m,s)}{4(S-1)}\le XM$$

This contradicts (3.6)

If y_m satisfies case (II), then $\lim_{m\to\infty}x_m=0$ from (2.1) and Lemma (2.2). The proof is complete.

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