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A STUDY OF LAPLACE TRANSFORMS IN ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The Laplace transform is one of the most powerful analytical tools for solving linear ordinary differential equations (ODEs) with constant coefficients, particularly initial value problems arising in engineering and applied science. This paper presents a systematic study of the Laplace transform method, beginning with its formal definition, existence conditions, and fundamental operational properties, followed by a compiled table of standard transform pairs. A step-by-step methodology for converting a linear ODE into an algebraic equation in the complex frequency domain, solving for the transformed variable, and recovering the time-domain solution through inverse transformation and partial-fraction expansion is developed and illustrated with worked examples drawn from mechanical vibration and electrical circuit problems. Numerical simulations are used to visualize the underdamped, critically damped, and overdamped responses of a second-order system, the unit-step response of a forced system, and the transient current in a series RL circuit. A comparative analysis contrasts the Laplace transform approach with classical methods such as undetermined coefficients and variation of parameters, highlighting its advantages in handling discontinuous and impulsive forcing functions and in directly incorporating initial conditions. The results confirm that the Laplace transform offers a computationally efficient and conceptually unifying framework for analyzing linear time-invariant systems, with direct relevance to control theory, circuit analysis, and structural dynamics.

Keywords: Laplace transform, ordinary differential equations, initial value problems, inverse Laplace transform, s-domain, transfer function, partial fraction expansion.

1. Introduction

Ordinary differential equations (ODEs) form the mathematical backbone of models describing dynamic systems in physics, engineering, biology, and economics. Whenever a system's rate of change depends on its current state — a charging capacitor, a vibrating beam, a decaying radioactive sample — the governing law is naturally expressed as

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a differential equation. While classical solution techniques such as the method of undetermined coefficients and variation of parameters are effective for simple forcing functions, they become cumbersome when the forcing term is discontinuous, periodic, or impulsive, and they do not incorporate initial conditions until the final step of the solution process.

The Laplace transform, named after Pierre-Simon Laplace, converts a linear differential equation in the time domain into an algebraic equation in a complex frequency domain, commonly denoted by the variable s. This transformation replaces differentiation with multiplication and integration with division, reducing the problem of solving a differential equation to that of solving an algebraic one. Initial conditions are absorbed directly into the transformed equation, and once the algebraic equation is solved for the transformed function, the inverse Laplace transform restores the solution to the time domain.

This paper aims to present a self-contained study of the Laplace transform as applied to ODEs. Section II reviews relevant background. Section III introduces the mathematical definition and properties of the transform. Section IV describes the general solution methodology. Section V presents worked examples with supporting figures. Section VI surveys engineering applications, Section VII offers a comparative analysis against classical methods, Section VIII discusses the results, and Section IX concludes the paper.

2. Literature Review:

The operational approach to differential equations traces its roots to the symbolic calculus developed by Oliver Heaviside in the late nineteenth century for analyzing electrical circuits. Although Heaviside's methods were effective, they lacked rigorous mathematical justification. It was later shown that Heaviside's operational calculus could be placed on a firm footing using the integral transform originally studied by Laplace, giving rise to what is now known as the Laplace transform method.

Standard treatments of the Laplace transform appear in nearly every textbook on engineering mathematics and differential equations, where it is presented as a bridge between classical analysis and operational methods. In control theory, the Laplace transform underlies the concept of the transfer function, which describes the input-output relationship of a linear time-invariant system in the s-domain. In circuit analysis, it allows resistive, inductive, and capacitive elements to be represented by algebraic impedances, greatly simplifying transient analysis. More recent work has extended Laplace-based techniques to fractional-order differential equations and to numerical inversion algorithms for cases where a closed-form inverse is not readily available.

3. Mathematical Preliminaries:

A. Definition of the Laplace Transform

Let f(t) be a function defined for $t \geqslant 0$. The Laplace transform of f(t), denoted $\mathscr{L}\{f(t)\}$ or F(s), is defined as

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$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \qquad (1)$$

The integral converges provided f(t) is piecewise continuous on every finite interval [0, T] and is of exponential order, meaning there exist constants M > 0 and α such that |f(t)| ⩽ M e^{α t} for all t ⩾ 0. Under these conditions, F(s) exists for all s > α.

B. Properties of the Laplace Transform

The utility of the Laplace transform in solving ODEs stems from a small set of operational properties, summarized in Table I. The differentiation property is of particular importance because it converts the derivative of a function into an algebraic expression involving F(s) and the initial conditions, which is precisely what allows an ODE to be reduced to an algebraic equation.

Table 1: Fundamental Properties of the Laplace Transform

Property	Time Domain	s-Domain
Linearity	a f(t) + b g(t)	a F(s) + b G(s)
First derivative	f′(t)	sF(s) − f(0)
Second derivative	f″(t)	$s^2F(s) - sf(0) - f'(0)$
Integration	$\int_0^t f(\tau)\, d\tau$	F(s)/s
Frequency shift	e^{at} f(t)	F(s − a)
Time shift	f(t − a)u(t − a)	e^{−as} F(s)
Scaling	f(at)	(1/a) F(s/a)
Convolution	(f * g)(t)	F(s) G(s)

C. Common Laplace Transform Pairs

Solving an ODE by the Laplace transform method requires, in the final step, an inverse transformation from the s-domain back to the time domain. This is typically accomplished by expressing F(s) as a sum of simpler terms through partial fraction expansion and then matching each term against a table of known transform pairs. Table II lists the pairs most frequently required when solving linear ODEs with constant coefficients.

Table 2: Common Laplace Transform Pairs

f(t)	F(s)
1	1/s
t	$1/s^2$
t^n (n = 0,1,2,...)	$n!/s^{n+1}$
e^{at}	1/(s − a)
sin(ωt)	$\omega/(s^2 + \omega^2)$
cos(ωt)	$s/(s^2 + \omega^2)$
e^{at} sin(ωt)	$\omega/[(s-a)^2 + \omega^2]$
e^{at} cos(ωt)	$(s-a)/[(s-a)^2 + \omega^2]$
u(t − a) (unit step)	e^{−as}/s
δ(t − a) (Dirac delta)	e^{−as}

4. Methodology

The general procedure for solving a linear ODE with constant coefficients using the Laplace transform consists of four steps, illustrated schematically in Fig. 1.

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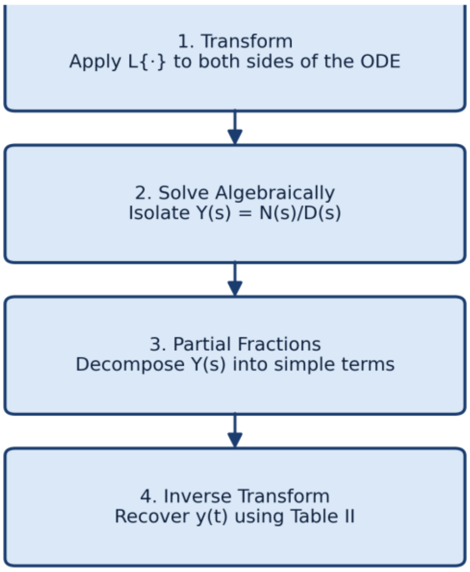

Fig 1: Flowchart of the four-step Laplace transform method for solving a linear ODE.

Step 1: Transform: Apply the Laplace transform to both sides of the ODE. Using the differentiation property,

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) \quad (2)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) \quad (3)$$

The equation is converted into an algebraic expression in Y(s) that already incorporates the initial conditions y(0) and y′(0).

Step 2: Solve algebraically: Rearrange the transformed equation to isolate Y(s), typically of the form.

$$Y(s) = N(s) / D(s) \quad (4)$$

Where N(s) and D(s) are polynomials in s determined by the coefficients of the ODE, the initial conditions, and the transform of the forcing function.

Step 3: Partial fraction expansion: Decompose Y(s) into a sum of simpler rational terms whose inverse transforms are known, according to the nature of the roots of D(s) (real distinct, repeated, or complex conjugate).

Step 4: Inverse transform: Apply Table II term-by-term to recover y(t), the solution of the original ODE in the time domain..

5. Illustrative Examples

A. Example 1: Free Vibration of a Second-Order System

Consider the homogeneous ODE $m y'' + c y' + k y = 0$, normalized to the standard form

$$y'' + 2\zeta\omega_n y' + \omega_n^2 y = 0, \; y(0) = 1, \; y'(0) = 0 \quad (5)$$

Where ω_n is the undamped natural frequency and ζ is the damping ratio. Taking the Laplace transform and applying (2)-(3) gives

$$Y(s) = (s + 2\zeta\omega_n) / (s^2 + 2\zeta\omega_n s + \omega_n^2) \quad (6)$$

The nature of the inverse transform depends on the roots of the denominator, which correspond to the poles of Y(s) in the s-plane, plotted in Fig. 2 for ω_n = 2 rad/s and three values of ζ. For $\zeta < 1$ (underdamped), the poles are complex conjugates and the response oscillates while decaying exponentially; for $\zeta = 1$ (critically damped), the poles coincide on the real axis; and for $\zeta > 1$ (overdamped), the poles are real and distinct, producing a non-oscillatory decay. The corresponding time-

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domain responses, obtained by inverse Laplace transformation, are plotted in Fig. 3.

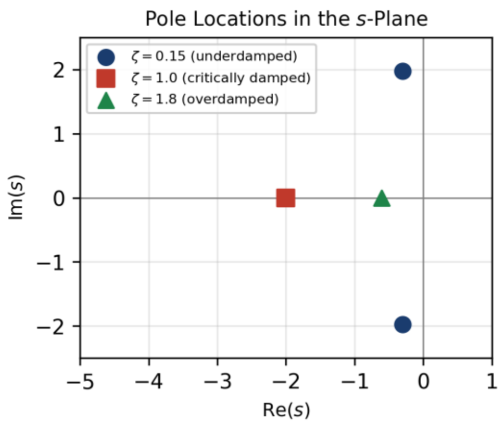

Fig 2: Pole locations in the s-plane for underdamped, critically damped, and overdamped cases.

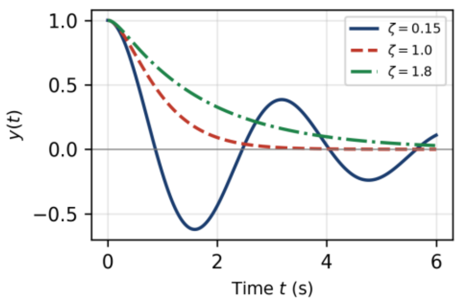

Fig 3: Time-domain response y(t) for underdamped, critically damped, and overdamped conditions, obtained via inverse Laplace transform.

B. Example 2: Forced Response to a Unit Step Input

Consider the forced ODE

$$y'' + 4y' + 5y = 5u(t), \; y(0) = 0, \; y'(0) = 0 \qquad (7)$$

Where u(t) is the unit step function, for which $\mathscr{L}\{u(t)\} = 1/s$. Transforming (7) gives

$$Y(s) = 5 / [s(s^2 + 4s + 5)] \; (8)$$

Partial fraction expansion followed by inverse transformation yields the closed-form solution

$$y(t) = 1 - e^{-2t}(\cos t + 2 \sin t) (9)$$

Which is plotted in Fig. 4. The response settles to the steady-state value of 1, consistent with the final value theorem, lim(s→0) sY(s) = 1..

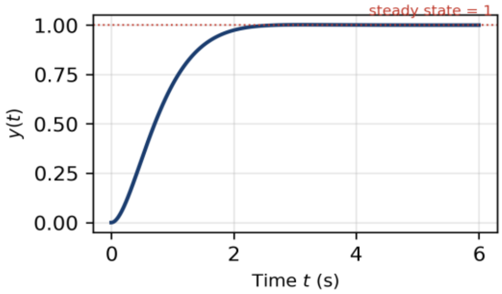

Fig 4: Unit-step response of the forced second-order system in (7), computed via Laplace transform.

C. Example 3: Transient Analysis of a Series RL Circuit

A series circuit containing a resistor R and an inductor L, connected to a constant voltage source V at t = 0, is governed by

$$L\,di/dt + R\,i = V, \; i(0) = 0 \qquad (10)$$

Applying the Laplace transform and solving for I(s) gives

$$I(s) = V / [s(Ls + R)] \qquad (11)$$

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Whose inverse transform is the classical exponential charging Response.

$$i(t) = (V/R)(1 - e^{-Rt/L}) \quad (12)$$

For R = 5 Ω, L = 2 H, and V = 10 V, the current response is plotted in Fig. 5, reaching approximately 63% of its final value at one time constant, τ = L/R = 0.4 s.

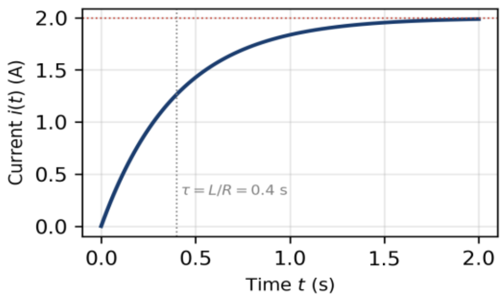

Fig 5: Transient current response of a series RL circuit computed using the Laplace transform.

6. Applications in Engineering and Science

The three examples in Section V are representative of a much broader class of problems in which the Laplace transform is the method of choice. In mechanical engineering, it is used to analyze the vibration of spring-mass-damper systems, suspension systems, and structural elements subjected to impact or periodic loading. In electrical engineering, it forms the basis for transient and steady-state analysis of RLC networks and underlies the transfer-function formulation used throughout classical control theory, where the ratio Y(s)/X(s) of output to input transforms characterizes system behavior independent of the specific input applied.

In chemical and thermal engineering, Laplace-based solutions describe heat conduction and the response of well-mixed reactor systems to changes in feed conditions. Beyond continuous physical systems, the transform's ability to handle discontinuous and impulsive inputs — through the unit step and Dirac delta functions — makes it well suited to modeling switching events, such as a circuit breaker closing or a sudden impact force, without requiring the ODE to be solved piecewise.

7. Comparative Analysis

Table 3 compares the Laplace transform method against two classical analytical techniques commonly taught alongside it: the method of undetermined coefficients and variation of parameters.

Table 3: Comparison of Solution Methods for Linear ODEs

Criterion	Laplace Transform	Undetermined Coeff.	Variation of Parameters
Handles discontinuous/impulsive forcing	Yes, directly	No	Limited
Initial conditions	Incorporated at start	Applied at end	Applied at end
Applicable coefficient type	Constant	Constant	Constant or variable
Applicable forcing type	Broad (transformable)	Polynomial/exp./trig.	Any continuous f(t)
Computational effort (high order)	Moderate (partial fractions)	High	High
System-level insight (poles, stability)	Direct (s-plane)	Indirect	Indirect

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8. Results and Discussion:

The worked examples confirm several practical advantages of the Laplace transform method. First, initial conditions are incorporated automatically at the transformation step, in contrast with classical methods where the general solution must first be found and the initial conditions applied afterward through a separate system of equations. Second, discontinuous and impulsive forcing functions, which are difficult to accommodate with undetermined coefficients, are handled directly through the transforms of the unit step and Dirac delta functions. Third, the pole locations of Y(s), visualized in Fig. 2, provide immediate qualitative insight into system behavior — oscillatory versus non-oscillatory, stable versus unstable — without requiring the time-domain solution to be computed explicitly, which is the foundation of pole-zero analysis in control system design.

The method does have limitations. It is most directly applicable to linear ODEs with constant coefficients; for equations with variable coefficients or nonlinear terms, closed-form Laplace transforms of the relevant terms may not exist, and numerical methods such as Runge-Kutta integration are generally preferred. Additionally, when D(s) in (4) has high-order or numerically obtained roots, partial fraction expansion becomes tedious, motivating the use of numerical inverse Laplace transform algorithms in practice. Nevertheless, for the broad class of linear time-invariant systems encountered in engineering, the Laplace transform remains a computationally efficient and conceptually unifying approach.

9. Conclusion

This paper presented a structured study of the Laplace transform as a method for solving linear ordinary differential equations. The definition, existence conditions, and key operational properties of the transform were reviewed, and a general four-step solution methodology was developed and demonstrated on three representative problems: free vibration of a damped second-order system, the forced step response of the same system, and the transient current in a series RL circuit.

Numerical simulations illustrated the effect of the damping ratio on system response and confirmed the correspondence between s-plane pole locations and time-domain behavior. A comparative analysis showed that the Laplace transform offers clear advantages over classical solution methods when initial conditions and discontinuous forcing functions are involved. Future work may extend this study to systems of coupled ODEs, fractional-order differential equations, and numerical inverse Laplace transform algorithms for cases lacking closed-form solutions.

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